Engineering Analysis with Boundary Elements 40 (2014) 71–77

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Analysis of piezoelectric plates with a hole using nature boundary integral equation and domain decomposition

Xing-Yuan Miao, Guo-Qing Li*

School of Civil Engineering and Mechanics, Huazhong University of Science and Technology, Wuhan 430074, China

ARTICLE INFO

Article history:
Received 5 May 2013
Accepted 18 November 2013
Available online 22 December 2013

Keywords:
Piezoelectric medium
Boundary integration equation
Harmonic analysis
Defects
Domain decomposition

ABSTRACT

In this paper, the plane problems of piezoelectricity are studied by using nature boundary integral equation and domain decomposition. A general displacement solution in terms of three potential functions is adopted to solve exterior boundary value problems of piezoelectricity, and three mapping relations corresponding to three potential functions are proposed for domain decomposition. By symbolic matrix inversion and derivation calculus, each potential function is governed by harmonic second-order partial differential equation in transformed domain with prescribed boundary condition. Therefore, three classic harmonic problems equivalent to the original plane piezoelectricity are established. Two cases of boundary conditions are considered, in which the displacement and electric potential are prescribed or the traction and electric displacement are given on the boundary. All problems considered are equivalent to three independent harmonic problems, which are solved by using nature boundary integration method proposed by Feng and Yu. A piezoelectric plate with a circular hole is analyzed as numerical examples. The results show that the proposed method is valid for the piezoelectric plates with holes. The proposed method has potential applications to analyze multi-field coupling problems.

1. Introduction

The study on coupled fields involved with piezoelectricity, electrostriction and magnetostriction has aroused many researchers' interests, and the plane problems of piezoelectricity have been widely and deeply investigated in the past years. Sosa and Castro [16] first studied the concentrated loads acted on the boundary of a piezoelectric half plane and extended the complex potential function method proposed by Lekhnitskii to analyze electroelastic problems, who also investigated the plane problems in piezoelectric medium with elliptic hole [15,17]. Ding et al. [3] derived the general solution of plane problems of piezoelectric medium expressed by harmonic functions. Rajapakse [14] investigated the upper half plane problem of piezoelectric medium by means of the Fourier transform. Furthermore, Benveniste [1], Chung and Ting [2], Dunn and Wienecke [7], Lu et al. [11], Pan and Yuan [13], Pan and Tonon [12] and Hu et al. [8] studied the Green's functions for a series of numerical computations of plane problems. Li et al. [10] analyzed the piezoelectric plane problem with fixed electrodes. As for computing methods, Yu and Zhao [22] and Wu and Yu [19] applied the natural boundary integral equation in many numerical experiments based on natural boundary reduction. Natural BIE has unique superiorities in multiple boundary value problems. In recent years, the natural BIE has been applied to solve a series of exterior problems for continuous or discrete domain in two-dimensional space. It can be referred to Wu and Yu [19], Du an Yu [5], Yu [21], Du and Yu [6], Yu and Zhao [22], Huang et al. [9] for more details.

In this paper, a coupling problem is decomposed into three succinct harmonic problems. First, the piezoelectric plane problem with specified displacement and electric potential boundary conditions will be investigated. The general solutions of displacement and electric potential are expressed by three harmonic functions. Then, we separate the boundary conditions into three mapping regions, and eventually the original problem would be transformed into harmonic problem in three different mapping regions. Second, we discuss piezoelectric plane problem of the second kind with specified distributed force and electric displacement boundary conditions. The general solutions of the stress and electric displacement are denoted by three harmonic functions. Hence, the equivalent three harmonic problems have been obtained by means of separating the boundary conditions as well.

2. General solution to plane piezoelectricity

In this section, a general solution to plane piezoelectricity using displacement potential functions will be briefly introduced, which is first proposed by Ding et al. [4]. Considering the piezoelectric medium occupying a plane without body force and electric charge,

* Corresponding author: Tel.: +86 13971016840; fax: +86 027 87543538.
E-mail address: lig57@hust.edu.cn (G.-Q. Li).

0955-7997/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.enganabound.2013.11.012

the governing equations for this piezoelectric problem are given as follows:

$$D\{u, w, \phi\}^T = 0, \tag{1}$$

where

$$D = \begin{bmatrix} c_{11}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2} & (c_{13}+c_{44})\frac{\partial^2}{\partial x_1 x_3} & (e_{15}+e_{31})\frac{\partial^2}{\partial x_1 x_3} \\ (c_{13}+c_{44})\frac{\partial^2}{\partial x_1 x_3} & c_{44}\frac{\partial^2}{\partial x_1^2} + c_{33}\frac{\partial^2}{\partial x_3^2} & e_{15}\frac{\partial^2}{\partial x_1^2} + e_{33}\frac{\partial^2}{\partial x_3^2} \\ -(e_{15}+e_{31})\frac{\partial^2}{\partial x_1 x_3} & -\left(e_{15}\frac{\partial^2}{\partial x_1^2} + e_{33}\frac{\partial^2}{\partial x_3^2}\right) & \varepsilon_{11}\frac{\partial^2}{\partial x_1^2} + \varepsilon_{33}\frac{\partial^2}{\partial x_3^2} \end{bmatrix}. \tag{2}$$

In the above equations, u and w are the mechanical displacement components along x_1 and x_3 directions, ϕ is the electric potential, D represents a differential matrix operator. $C_{ij}, e_{ij}, \varepsilon_{ij}$ are the elastic, piezoelectric and dielectric constants, respectively.

Based on the derivation in Ding's work [4], the fundamental solution to Eq. (1) is expressed in terms of three quasi-harmonic functions:

$$\left(\frac{\partial^2}{\partial y_{k1}^2} + \frac{\partial^2}{\partial y_{k3}^2}\right)\psi_k = 0, \tag{3}$$

where subscript $k = 1, 2, 3$, which will be always implicit implied in this paper, and ψ_k are potential functions, $y_{k1} = x_1 + \alpha_k x_3$, $y_{k3} = \beta_k x_3$ and $s_k = \alpha_k + \mathrm{i}\beta_k$ are the three roots of the following equation:

$$as^6 - bs^4 + cs^2 - d = 0, \tag{4}$$

where

$$\begin{aligned} a &= c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}), \\ b &= c_{33}[c_{44}\varepsilon_{11} + (e_{15}+e_{31})^2] + \varepsilon_{33}[c_{11}c_{33} + c_{44}^2 - (c_{13}+c_{44})^2] \\ &\quad + e_{33}[2c_{44}e_{15} + c_{11}e_{33} - 2(c_{13}+c_{44})(e_{15}+e_{31})], \\ c &= c_{44}[c_{11}\varepsilon_{33} + (e_{15}+e_{31})^2] + \varepsilon_{11}[c_{11}c_{33} + c_{44}^2 - (c_{13}+c_{44})^2] \\ &\quad + e_{15}[2c_{11}e_{33} + c_{44}e_{15} - 2(c_{13}+c_{44})(e_{15}+e_{31})], \\ d &= c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}). \end{aligned} \tag{5}$$

It has been studied on s_k by Ding et al. [4] and by [18] that, the eigenvalue s_k has a positive imaginary part, i.e. $\beta_k > 0$, and s_k can also be a pure imaginary number. For simplicity, we only consider the s_k as imaginary in this paper. In the rest of this paper, $y_{k1} = x_1$, $y_{k3} = y_k = s_k x_3$ will be always established.

Therefore, the displacement components and electric potential can be obtained as

$$u = \sum_{k=1}^{3} a_{1k}\frac{\partial \psi_k}{\partial x_1}, \quad w = \sum_{k=1}^{3} a_{2k}\frac{\partial \psi_k}{\partial y_k}, \quad \varphi = \sum_{k=1}^{3} a_{3k}\frac{\partial \psi_k}{\partial y_k}, \tag{6}$$

where

$$\begin{aligned} a_{1k} &= 1, \\ a_{2k} &= \frac{(c_{11}\varepsilon_{11} - m_3 s_k^2 + c_{44}\varepsilon_{33}s_k^4)}{(m_1 - m_2 s_k^2)s_k}, \\ a_{3k} &= \frac{(c_{11}e_{15} - m_4 s_k^2 + c_{44}e_{33}s_k^4)}{(m_1 - m_2 s_k^2)s_k}, \end{aligned} \tag{7}$$

in which

$$\begin{aligned} m_1 &= (c_{13}+c_{44})\varepsilon_{11} + (e_{15}+e_{31})e_{15}, \\ m_2 &= (c_{13}+c_{44})\varepsilon_{33} + (e_{15}+e_{31})e_{33}, \\ m_3 &= c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15}+e_{31})^2, \\ m_4 &= c_{11}e_{33} - c_{13}(e_{15}+e_{31}) - c_{44}e_{31}. \end{aligned} \tag{8}$$

Applying the constitutive equations of the plane piezoelectricity to the displacement solution, the stress components and electric displacement can be obtained by

$$\sigma_{11} = \sum_{k=1}^{3} b_{11}\frac{\partial^2 \psi_k}{\partial x_3^2}, \quad \sigma_{33} = \sum_{k=1}^{3} b_{12}\frac{\partial^2 \psi_k}{\partial x_1^2}, \quad \sigma_{13} = \sum_{k=1}^{3} b_{13}\frac{\partial^2 \psi_k}{\partial x_1 \partial x_3}, \tag{9}$$

$$D_1 = \sum_{k=1}^{3} b_{14}\frac{\partial^2 \psi_k}{\partial x_1^2}, \quad D_3 = \sum_{k=1}^{3} b_{15}\frac{\partial^2 \psi_k}{\partial y_k^2}, \tag{10}$$

where

$$\left.\begin{aligned} b_{11} &= C_{11}a_{1k} + C_{13}a_{2k} + e_{31}a_{3k}, \\ b_{12} &= C_{13}a_{1k} + C_{33}a_{2k} + e_{33}a_{3k}, \\ b_{13} &= C_{44}(a_{1k} + a_{2k}) + e_{15}a_{3k}, \\ b_{14} &= e_{15}(a_{1k} + a_{2k}) - \varepsilon_{11}a_{3k}, \\ b_{15} &= e_{31}a_{1k} + e_{33}a_{2k} - \varepsilon_{33}a_{3k}. \end{aligned}\right\} \tag{11}$$

In summary, there are three harmonic functions ψ_k in transformed domain $\Omega_k(x_1, y_k)$ satisfying the governing equation of plane piezoelectricity, and all physical fields can be expressed in terms of the functions ψ_k. In order to obtain the solution of ψ_k fulfilling the prescribed boundary equations, the equations on $\partial\Omega(x_1, x_3)$ will be separated into three independent boundary conditions on the transformed boundary $\partial\Omega_k(x_1, y_k)$, which will be presented in the following sections.

3. Plane piezoelectricity with displacement and electric potential conditions

In this section, the displacement and electric potential boundary conditions will be spliced into three independent conditions in transformed domain.

It is assumed that the original conditions for static elasto-electric are given as follows:

$$u|_{\partial\Omega} = u_0(x_1, x_3), \quad w|_{\partial\Omega} = w_0(x_1, x_3), \quad \varphi|_{\partial\Omega} = \varphi_0(x_1, x_3). \tag{12}$$

Substituting Eq. (6) into Eq. (12) yields the boundary conditions for the displacement and electric potential, that is

$$\begin{aligned} &\sum_{k=1}^{3} a_{1k}\frac{\partial \psi_k}{\partial x_1} = u_0(x_1, x_3), \\ &\sum_{k=1}^{3} a_{2k}\frac{\partial \psi_k}{\partial y_k} = w_0(x_1, x_3), \\ &\sum_{k=1}^{3} a_{3k}\frac{\partial \psi_k}{\partial y_k} = \varphi_0(x_1, x_3). \end{aligned} \tag{13}$$

Considering a simply connected region surrounded by one smooth curve, which can be defined by the following t-parametric equations:

$$x_1 = f_1(t), \quad x_3 = f_3(t), \tag{14}$$

For purpose of separating the boundary conditions into the transformed domains, the coordinate transformation relation is used. Then, Eq. (13) can be rewritten with respect to t-parameter as follows:

$$\begin{aligned} &\sum_{k=1}^{3} \frac{a_{1k}}{f_1'}\frac{\partial \psi_k}{\partial t} = u_0(x_1, x_3), \\ &\sum_{k=1}^{3} \frac{a_{2k}}{s_k f_3'}\frac{\partial \psi_k}{\partial t} = w_0(x_1, x_3), \\ &\sum_{k=1}^{3} \frac{a_{3k}}{s_k f_3'}\frac{\partial \psi_k}{\partial t} = \varphi_0(x_1, x_3). \end{aligned} \tag{15}$$

Rewriting Eq. (15) into matrix form leads to

$$\mathbf{A}(t)\frac{\partial \psi_k}{\partial t} = \mathbf{U}(t), \tag{16}$$