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Stable numerical solution to a Cauchy problem for a time fractional diffusion equation ☆

T. Wei *, Z.Q. Zhang

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, Gansu Province, PR China

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ABSTRACT

In this paper, we consider a Cauchy problem of one-dimensional time fractional diffusion equation for determining the Cauchy data at $x=1$ from the Cauchy data at $x=0$. Based on the separation of variables and Duhamel's principle, we transform the Cauchy problem into a first kind Volterra integral equation with the Neumann data as an unknown function and then show the ill-posedness of problem. Further, we use a boundary element method combined with a generalized Tikhonov regularization to solve the first kind integral equation. The generalized cross validation choice rule is applied to find a suitable regularization parameter. Three numerical examples are provided to show the effectiveness and robustness of the proposed method.

1. Introduction

Fractional diffusion equations have attracted wide attentions in recent years which can be used to describe anomalous diffusion phenomena instead of classical diffusion procedure. Time fractional diffusion equations are deduced by replacing the standard time derivative with a time fractional derivative and can be used to describe superdiffusion and subdiffusion phenomena [34,19,20,41,1,32]. Indeed, fractional diffusion equations have numerous applications in nanotechnology, viscoelasticity, hereditary solid mechanics, chemistry and biochemistry, signal and image processing, and other fields in engineering [2]. The direct problems, i.e., initial value problem and initial boundary value problems for time fractional diffusion equations have been studied extensively in recent years, for instances, on maximum principle [17], on some uniqueness and existence results [16,31], on numerical solutions by finite element methods [8] and finite difference methods [40,14,22,33], on exact solutions [38,19,18].

However, in some practical situations, a part of boundary data, or initial data, or diffusion coefficients, or source term may not be given and we want to find them by additional measured data which will give rise to some fractional diffusion inverse problems. The early papers on inverse problems were provided by Murio in [21,23,24] for solving the sideways fractional heat equations by mollification methods. After that, some works on fractional inverse problems have been published. In [3], Cheng et al. considered an inverse problem for determining the order of fractional derivative and diffusion coefficient in a fractional diffusion equation and gave a uniqueness result. In [15], Liu et al. solved a backward problem for the time-fractional diffusion equation by a quasi-reversibility regularization method. Zheng et al. in [43,44] solved the Cauchy problems for the time fractional diffusion equations on a strip domain by a Fourier truncation method and a convolution regularization method. Qian in [29] used a modified kernel method to deal with a sideways fractional equation inverse problem. In [42,25,31,37,4,11,12], some inverse source problems for various fractional diffusion equations were investigated. Furthermore, the nonlinear fractional inverse problems have been considered recently in [30,9]. To our knowledge, the study of inverse problems for fractional differential equations is in a very early developmental stage.

In this paper, we consider the following Cauchy problem for a time fractional diffusion equation:

$$\begin{cases} {}_0\partial_t^\alpha u = u_{xx}, & 0<x<1,\ 0<t<T, \\ u(0,t)=h(t), & 0\le t\le T, \\ u_x(0,t)=q(t), & 0\le t\le T, \\ u(x,0)=\varphi(x), & 0\le x\le 1, \end{cases} \tag{1.1}$$

where u is the solute concentration and ${}_0\partial_t^\alpha u$ is the Caputo time-fractional derivative of order α defined in [27] by

$${}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}, \quad 0<\alpha<1, \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function. The main aim in this study is to recover the Dirichlet data $u(1,t)$ and the Neumann data $u_x(1,t)$ in the finite time period $[0,T]$.

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* Corresponding author. Tel.: +86 931 8912493; fax: +86 931 8912481.
E-mail address: tingwei@lzu.edu.cn (T. Wei).

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For this Cauchy problem, one can suppose that the solute concentration and spread velocity of pollution in soil at one end of a finite distance are not available to be measured since this end is far away or inaccessible, however the data of solute concentration and spread velocity can be obtained at another end. Our propose is to determine the missing data on the inaccessible end from the measured data on the accessible end. This is the physical motivation for considering the Cauchy problem (1.1).

The Cauchy problem mentioned above is an ill-posed problem, refer to Remark 2.7 in Section 2. That means the solution (if exists) does not depend continuously on the given data and any small perturbation in the given data may cause large change to the solution. However under an additional priori assumption to the solution, Xu et al. in [39] proved firstly a conditional stability for a special fractional derivative order $\alpha=1/2$. From which we know the solution of problem (1.1) is unique for this special case $\alpha=1/2$. As for a general $\alpha\in(0,1)$, the uniqueness is still open.

In this paper, we focus on numerical solution for problem (1.1) with a general order α. A generalized Tikhonov regularization method based on a boundary element method is used to determine the Neumann data and the Dirichlet data at $x=1$.

Our paper is divided into five sections. In Section 2, we transform the Cauchy problem (1.1) into a Volterra boundary integral equation by separation of variables and Duhamel's principle. In Section 3, we propose a regularized method based on boundary element discretization for recovering a stable approximation to $u_x(1,t)$. The computational formulation for recovering $u(1,t)$ is also provided. Numerical experiments for three examples are investigated in Section 4. Finally, we give a conclusion in Section 5.

2. Boundary integral equation

Throughout this paper, we use the following definitions and propositions, see [10,27].

Definition 2.1. Let $z(t)$ be absolutely continuous on $[a,b]$. For $0<\alpha<1$, the Caputo fractional derivative ${}_a\partial_t^\alpha z(t)$ and the Riemann–Liouville fractional derivative ${}_aD_t^\alpha z(t)$ can be defined in the forms

$$ {}_a\partial_t^\alpha z(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{z'(s)}{(t-s)^\alpha}\,ds,\quad a<t<b, \tag{2.1}$$

and

$$ {}_aD_t^\alpha z(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_a^t\frac{z(s)}{(t-s)^\alpha}\,ds,\quad a<t<b, \tag{2.2}$$

respectively.

Proposition 2.2. *Let $z(t)$ be absolutely continuous on $[a,b]$. Then the Caputo fractional derivative ${}_a\partial_t^\alpha z(t)$ and the Riemann–Liouville fractional derivative ${}_aD_t^\alpha z(t)$ exist almost everywhere on $[a,b]$, there is a relationship between the Caputo fractional derivative and the Riemann–Liouville fractional derivative*

$$ {}_aD_t^\alpha z(t)=\frac{1}{\Gamma(1-\alpha)}\frac{z(a)}{(t-a)^\alpha}+{}_a\partial_t^\alpha z(t),\quad a<t<b.$$

Lemma 2.3. *For $0<\alpha<1$ and $f(t)\in C^1[0,T]$, we have*

$$ {}_0D_t^{1-\alpha}\,{}_0\partial_t^\alpha f(t)=f'(t). \tag{2.3}$$

Definition 2.4. The generalized Mittag–Leffler function is defined by

$$ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{2.4}$$

where $\alpha>0$, $\beta\in\mathcal{R}$.

Proposition 2.5. *Let $\lambda>0$, then we have*

$$ \frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha)=-\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha),\quad t>0,\ \alpha>0. \tag{2.5}$$

Based on the separation of variables, we can deduce a first kind Volterra boundary integral equation satisfied by the Neumann data at $x=1$.

Denote $f(t):=u_x(1,t)$. Let $v(x,t)=u(x,t)-((x^2/2)f(t)+(x(2-x)/2)q(t))$, then by a simple calculation, we know that the function $v(x,t)$ satisfies the following direct problem of time-fractional diffusion equation:

$$\begin{aligned}&{}_0\partial_t^\alpha v-v_{xx}=g(x,t),\quad 0<x<1,\ 0<t<T,\\&v_x(0,t)=0,\quad 0\le t\le T,\\&v_x(1,t)=0,\quad 0\le t\le T,\\&v(x,0)=\phi(x),\quad 0\le x\le 1,\end{aligned}$$

where

$$ g(x,t)=-\left(\frac{x^2}{2}{}_0\partial_t^\alpha f(t)+\frac{x(2-x)}{2}{}_0\partial_t^\alpha q(t)\right)+f(t)-q(t) \tag{2.6}$$

and

$$ \phi(x)=\varphi(x)-\left(\frac{x^2}{2}f(0)+\frac{x(2-x)}{2}q(0)\right).$$

Suppose $v(x,t)=W(x,t)+P(x,t)$ such that W and P satisfy the following problem (2.7) and (2.8), respectively:

$$\begin{cases}{}_0\partial_t^\alpha W-W_{xx}=0, & 0<x<1,\ 0<t<T,\\ W_x(0,t)=0, & 0\le t\le T,\\ W_x(1,t)=0, & 0\le t\le T,\\ W(x,0)=\phi(x), & 0\le x\le 1.\end{cases} \tag{2.7}$$

$$\begin{cases}{}_0\partial_t^\alpha P-P_{xx}=g(x,t), & 0<x<1,\ 0<t<T,\\ P_x(0,t)=0, & 0\le t\le T,\\ P_x(1,t)=0, & 0\le t\le T,\\ P(x,0)=0, & 0\le x\le 1.\end{cases} \tag{2.8}$$

By the separation of variables, the solution of problem (2.7) can be written formally as an infinite series (refer to [42])

$$ W(x,t)=\sum_{k=0}^{\infty}\phi_kE_{\alpha,1}(-k^2\pi^2t^\alpha)\cos(k\pi x), \tag{2.9}$$

where ϕ_k are the Fourier coefficients given by

$$ \phi_k=2\int_0^1\cos(k\pi x)\phi(x)\,dx,\ k\ge 1\quad\text{and}\quad \phi_0=\int_0^1\phi(x)\,dx, \tag{2.10}$$

and $E_{\alpha,1}$ is the Mittag–Leffler function defined in (2.4).

By using Duhamel's principle for a fractional diffusion equation, see Proposition 3 in [42] or the original one in [35], the solution of problem (2.8) can be expressed by

$$ P(x,t)=\int_0^tV(x,t;\tau)\,d\tau, \tag{2.11}$$

where $V(x,t;\tau)$ is the solution of following problem:

$$\begin{cases}{}_\tau\partial_t^\alpha V(x,t;\tau)=V_{xx}(x,t;\tau),(x,t)\in(0,1)\times(\tau,T), & \\ V(x,t;\tau)|_{t=\tau}={}_0D_\tau^{1-\alpha}g(x,\tau), & 0\le x\le 1,\\ V_x(1,t;\tau)=0, & \tau\le t\le T,\\ V_x(0,t;\tau)=0, & \tau\le t\le T,\end{cases} \tag{2.12}$$

in which ${}_0D_\tau^{1-\alpha}$ is the Riemann–Liouville fractional derivative defined in (2.2).

In the following, by using Lemma 2.3, we give a clear expression to the solution P of problem (2.8). From the definition of g in