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Numerical computation for backward time-fractional diffusion equation

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ABSTRACT

Based on kernel-based approximation technique, we devise in this paper an efficient and accurate numerical scheme for solving a backward problem of time-fractional diffusion equation (BTFDE). The kernels used in the approximation are the fundamental solutions of the time-fractional diffusion equation which can be expressed in terms of the *M*-Wright functions. To stably and accurately solve the resultant highly ill-conditioned system of equations, we successfully combine the standard Tikhonov regularization technique and the L-curve method to obtain an optimal choice of the regularization parameter and the location of source points. Several 1D and 2D numerical examples are constructed to demonstrate the superior accuracy and efficiency of the proposed method for solving both the classical backward heat conduction problem (BHCP) and the BTFDE.

1. Introduction

In the last decades, the investigation of backward heat conduction problem (BHCP) arises from many branches of engineering sciences. For instance, there is a great demand on a good estimation of heat temperature and heat flux history from only spatially observed data during a heat propagation process. In general, transient heat conduction phenomena are governed by the parabolic heat conduction equation. If the initial temperature distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable from solving a well-posed problem. In reality, however, the boundary conditions are usually either missing or incomplete; and the temperature distribution data can only be collected at some specified time intervals. Although heat conduction process is very smooth, the process is irreducible. This means that the characteristic of the solution (for instance, the shape of the interior heat flow) may not be affected by the observed data. On the other hand, the heat conduction process has no finite propagation speed and thus an efficient non-destructive testing technique can be achieved at a comparably much lower cost. The lack of mathematical analysis and efficient algorithm, however, hinders the development of such low cost and efficient non-destructive testing technique.

The BHCP is in nature unstable because the unknown solution and its derivatives have to be determined from indirect observable data which contain measurement error. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the problem and the ill-conditioning of the resultant discretized system of equations. It is a typical ill-posed problem in the sense that solution of BHCP does not continuously depend on the final temperature data. In fact, any small change in the given final temperature data may induce enormous change in the solution. Stable approximation by using regularization techniques has been investigated by Han et al. [1], Muniz et al. [2], etc. Recently, the Method of Fundamental Solutions (MFS) have been used respectively by Hon and Li [3], Liu [4], Mera [5] and Wei and Wang [6] for solving the BHCP. It is well known that the accuracy of the MFS depends on a suitable placement of source points. Mera in [5] proposed to put the source points on a line below the initial time whereas Hon and Li in [3] gave an improved solution by placing the source points uniformly over both the temporal and spatial axes. In [6] the authors provided a new choice method for locating the source points from using the single layer heat potential.

Partial differential equations of fractional orders have recently become a focus of many research studies because of their various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. Fractional calculus in mathematics is a natural extension of integer-order calculus. It has been used for modeling many physical processes arisen from real-life problems, for instance, the modeling on the transport of passive tracers carried by fluid flows in a porous medium under groundwater hydrology.

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Studies of the complicated phenomena of the interstitial fluid flows are still under intensive researches and are particularly challenging for quantitative analyses and modeling. The time-fractional partial differential equation, obtained from the standard partial differential equation that replaces the time derivative by a fractional derivative, is related with the continuous-time random walk and is a model for anomalous diffusion in many applied fields such as diffusion processes of contaminants in porous media. Numerical methods for solving well-posed initial boundary value problems of fractional diffusion equation can be found from the recent works of Wen et al. [7], Brunner et al. [8], and Cuesta and Palencia [9]. To the knowledge of the authors, there are still very little theoretical and computational works on solving BTFDE. More recently, the investigation of the BTFDE problem have respectively performed by Liu and Yamamoto [10] with the quasi-reversibility method; Ren et al. [11] with the spectral truncation method; and Wang et al. [12] with the Tikhonov regularization method.

In this paper, based on the Kernel-Based Approximation (KBA), we devise an efficient and accurate numerical scheme for solving the backward problem of time-fractional diffusion equation and propose a new strategy to choose the location of source points through the solution expressed in terms of integrals of the Green's functions obtained from solving the Cauchy problem. To solve the highly ill-conditioned resultant system of linear equations, we adapt the use of the standard Tikhonov regularization technique with the L-curve method for an optimal regularization parameter. Numerical result indicates an improvement on both efficiency and accuracy for solving the BTFDE in comparing with the works on BHCP given in [3–6] and BTFDE in [6].

This paper is organized as follows. In Section 2 we consider an inverse problem of BTFDE with the fractional derivative defined in the sense of Caputo. The numerical scheme based on the kernel-based approximation is devised in Section 3. A new strategy for the location of source points is given. Numerical verification on the efficiency and accuracy of the proposed method for both 1D and 2D BTFDE problems is presented in Section 4. Finally, we conclude the paper in Section 5.

2. Backward time-fractional diffusion equation

Consider the following time-fractional diffusion equation:

$$\frac{\partial^\beta u(\mathbf{x},t)}{\partial t^\beta} = \nabla^2 u, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n,\ t \in (0,T), \tag{2.1}$$

where $\partial^\beta u(\mathbf{x},t)/\partial t^\beta$ denotes the fractional derivative of given order β with respect to the time variable t in the sense of Caputo defined in [13,14] as

$$\frac{\partial^\beta \varphi(t)}{\partial t^\beta} = \begin{cases} \dfrac{\partial^n \varphi(t)}{\partial t^n}, & \beta = n, \\ \dfrac{1}{\Gamma(n-\beta)} \int_0^t \frac{\varphi^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}}\, d\tau, & n-1 < \beta \le n. \end{cases} \tag{2.2}$$

In the case of slow diffusion, the values of β are taken to be $0<\beta<1$. Eq. (2.1) subject to the boundary condition:

$$u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega, \quad 0 < t < T, \tag{2.3}$$

and the final condition:

$$u(\mathbf{x},T) = g(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.4}$$

is called an inverse backward problem for the time-fractional diffusion equation (BTFDE) in which the unknown solution $u(\mathbf{x},t)$ for $x \in \Omega, 0 \le t < T$ has to be determined from the boundary measurement f and terminal time measurement g, which normally contain noises in real measurement. Recently, the Caputo derivative has been extensively investigated due to its adaptability in treating physical and engineering problems which require standard initial conditions [13,14].

Proofs on the existence and stability for the solution of this BTFDE problem (2.1), (2.3), (2.4) have been established in [15]. Numerical computation, however, is still very rare due to the introduction of fractional derivatives. Based on the kernel approximation method, we devise in this paper an efficient and effective numerical method to approximate the solution under noisy data f and g.

In practical applications, the data $f(\mathbf{x})$ and $g(\mathbf{x},t)$ are given only at some scattered discrete points $(x_i,t_i)_{i=1}^{m+n}$:

$$u(\mathbf{x}_i,t_i) = f(\mathbf{x}_i,t_i), \quad i=1,\ldots,m, \tag{2.5}$$

$$u(\mathbf{x}_i,T) = g(\mathbf{x}_i), \quad i=m+1,\ldots,m+n. \tag{2.6}$$

where $\{\mathbf{x}_i\}_{i=1}^m$, $\{t_i\}_{i=1}^m$ denote respectively the discrete spatial values on the boundary $\partial\Omega$ and temporal values in the interval $(0,T]$, $\{\mathbf{x}_i\}_{i=m+1}^{m+n}$ denotes the discrete spatial values in the domain Ω. These discrete points $(\mathbf{x}_i,t_i)_{i=1,\ldots,m}$, $(\mathbf{x}_i,T)_{i=m+1,\ldots,m+n}$ are also called *collocation points*.

The fundamental solutions of Eq. (2.1) for general β are given as [16]

$$G_\beta(\mathbf{x},t) = \frac{1}{2t^{\beta/2}} M_{\beta/2}\left(\frac{|\mathbf{x}|}{t^{\beta/2}}\right) H(t), \tag{2.7}$$

where $H(t)$ is the Heaviside function and $M_{\beta/2}(|\mathbf{x}|/t^{\beta/2})$ is the M-Wright function defined as

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\nu n + (1-\nu)]} = \frac{1}{2\pi i}\int_{Ha} e^{\sigma - z\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}}, \quad 0<\nu<1,$$

where *Ha* denotes the Hankel path.

For computational purpose, we consider in this paper the cases when $\nu = 1/2$ and $\nu = 1/3$ so that the following identities hold

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)_n \frac{z^{2n}}{(2n)!} = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right), \tag{2.8}$$

$$M_{1/3}(z) = \frac{1}{\Gamma(2/3)} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)_n \frac{z^{3n}}{(3n)!} - \frac{1}{\Gamma(1/3)} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)_n \frac{z^{3n+1}}{(3n+1)!} = 3^{2/3} Ai\left(\frac{z}{3^{1/3}}\right), \tag{2.9}$$

where *Ai* denotes the *Airy function*.

We note here that the M-Wright functions, which was introduced by Mainardi et al. [16], are special kinds of Wright functions:

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[\lambda n + \mu]} = \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^\mu}, \quad \lambda > -1, \mu \in \mathbb{C},$$

for $\lambda = -\nu,\ \mu = 1-\mu$, i.e.,

$$M_\nu(z) = W_{-\nu,1-\nu}(-z), \quad 0<\nu<1,$$

whose numerical computation involves the evaluation of an infinite series of complex integrals.

For illustration, some figures of the fundamental solutions to problem (2.1) for different β are displayed in Fig. 1 with their characteristics outlined for $\beta = 1/2$ and $1/3$ in Fig. 2. From these figures we can observe the so-called "memory effect" of fractional derivatives, which imposes some difficulties in obtaining accurate approximation from using classical numerical techniques.