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An Approach Based on Generalized Functions to Regularize Divergent Integrals

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ABSTRACT

This article addresses weakly singular, hypersingular integrals, which arise when the boundary integral equation (BIE) methods are used for 3-D potential theory problem solutions. An approach based on the theory of distributions and the application of the second Green theorem has been explored for the calculation of such divergent integrals. The divergent integrals have been transformed to a form that allows easy and uniform calculation of weakly singular and hypersingular integrals. For flat boundary elements (BE), piecewise constants and piecewise linear approximations, only regular integrals over the contour of the BE have to be evaluated. Furthermore, all calculations can be done analytically, so no numerical integration is required. In the case of 3-D, rectangular and triangular BE have been considered. The behavior of divergent integrals has been studied in the context that the collocation point moves to the contour of the BE.

1. Introduction

In recent years, many publications have been devoted to discussing BIE methods and their application to science and engineering because the BIE and the discrete analogy boundary element method (BEM) are very powerful tools for solving the mathematical problems that arise in science and engineering [1,2,7,17]. A historical account of the main stages of BIE and BEM development can be found in [9]. One of the main problems of the BIE solution is the calculation of divergent integrals as the numerical methods developed for regular integrals cannot be used and special methods have to be applied.

Many methods have been developed using a classical approach for calculating the divergent integrals of different types. Usually, divergent integrals of a different type require different methods for their mathematical interpretation and numerical calculation. An analysis of most known methods used for the treatment of the different divergent integrals can be found in several books [1,7,17,20,25,27,33] and review articles [8,18,22,26,34,40]. From these publications, it follows that a significant impact on the topic under consideration has been made by Cruse [12–15], Mukherjee [28–31], Atluri [10,11,32,35] and their coauthors. There are some additional references that may also be of interest [3,4]. This publication is not intended to be a review; therefore, we do not present a long list of publications on the topic and will not critically analyze the problem. Instead, we concentrate on problems directly related to the topic of this publication. We begin by making one historical remark: the method of regularization for singular and hypersingular integrals, which was developed by Guiggiani and coauthors (see [16] and references there), is very popular in the BEM community. Roughly speaking, the method consists of the following: A singularity is extracted from the divergent integral and divided into several parts. One part is relatively simple and contains singularities, while the other parts are regular. The regular parts are calculated using established methods, and the singular part is usually known and can be calculated. Such an approach is not new; it has long been used for divergent series calculations [21] and for calculating divergent integrals. For example, Kantorovich in [24] calculated 1-D divergent integrals with different types of singularities, including hypersingular integrals, and then extracted the divergent parts and calculated them analytically. Michlin extended this approach for the n-dimensional case and the theoretical foundation of that method [26,27]. Since that time, such methods have been used for divergent integral treatment. For example, we used it to derive a solution for the elastodynamic contact problems for cracked bodies in [36].

The classic approach for treating divergent integrals has one significant disadvantage; divergent integrals with a different singularity need different definitions, different theoretical justifications and different methods to be calculated. For example, weakly singular integrals are considered improper integrals, singular integrals are considered in the sense of the Cauchy principal value (*PV*), and hypersingular integrals have been considered by Hadamard to be finite part integrals (*FP*).

In modern mathematics, divergent integrals have a strong theoretical foundation based on the theory of generalized functions (distributions), which permits us to apply the same approach for divergent integrals as for different types of singularities. According to this theory, divergent integrals with any type of singularity can be considered to be functionals (generalized functions) defined in

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special functional spaces and on specially defined test functions [6]. We have shown in several publications that the approach based on the theory of distributions has not only theoretical meaning but is also an important application for practical calculations of divergent integrals.

In our previous publications [17–19,37–44], the approach based on the theory of distributions has been developed for the regularization of divergent integrals with different singularities that arise in BEM applications. We explore the approach presented in [5], which interprets definite integrals as distributions and applies them to the solution of problems of fracture mechanics in [36]. Then, this approach was further developed for the regularization of 2-D hypersingular integrals, which appear in static and dynamic problems of fracture mechanics in [43] and [44], respectively. Regularized formulae for different types of divergent integrals have been reported in [38,39,40–41]. Additional applications of the regularization method can be found in review articles [18,19,42] and in a book [17]. Further development of that approach and the application of the second Green's theorems in the sense of the theory of distributions have been described in [37,38]. The case of piecewise linear approximation has been considered in [39,40], while regularized formulae obtained in [38,40,41] permit us to transform weakly divergent singular, singular and hypersingular integrals over any polygonal area into regular integrals over the area contour. The developed approach can be applied to the regularization of a wide class of divergent integral regularization. In addition to hypersingular integrals, it is also suitable for the regularization of a variety of divergent integrals and any polynomial approximation.

In this paper, the approach based on the theory of distributions and Green's second theorem is developed and applied to the regularization of the divergent integral that appears in 2-D and 3-D potential theory problems solved by BIE methods. Generalized Gauss-Ostrogradski and Green theorems, which are applicable for the case of singular functions, have been obtained using methods presented in [5]. Then, the generalized second Green theorem was used for the development of the regularized formulae for divergent integral calculation. A special case of 2-D divergent integral regularization has been assessed for weakly and hypersingular integrals, and regular formulae for their calculation have been obtained. The weakly singular and hypersingular integrals for piecewise constant approximation have been considered for arbitrary convex polygons and for piecewise linear approximations that have been considered for rectangular and triangular BE. Using regularized formulae obtained here, divergent integrals were calculated for circular, quadratic and triangular areas. It is important to mention that in all presented equations, all calculations can be done analytically and no numerical integration is needed. The behavior of the divergent integrals was studied in the context of the collocation point being situated inside and outside of the BE and also when moving to the contour of the BE; the results are illustrated by corresponding diagrams created with *Mathematica* software.

2. Divergent integrals and boundary integral equations

Divergent integrals occur in various mathematical and engineering applications and can be of various types of singularity that exhibit different behavior in the vicinity of singular points. In this study, we concentrate our efforts on calculating the divergent integrals that appear in BIE when the corresponding system of integral equations is solved numerically using BEM. Usually, changing coordinates corresponding to divergent integrals can be presented as

$$J_k^{i,j} = \int_S \frac{x_1^i x_2^j \phi(x)}{r^k} dS \tag{1}$$

where $r=\sqrt{x_1^2+x_2^2+x_3^2}$ is the distance in Euclidian space, $x_1$and $x_2$are the coordinates of integration and function $\phi(\mathbf{x})$is regular.

As soon as the point that corresponds to the origin of the coordinated system belongs to Ω, the integral (1) is divergent. The type of singularity depend on the value of the indices i, j and k. Because of the aforementioned singularity integrals, (1) cannot be transformed using the regular Green's theorem. Therefore, a generalized function approach will be used here.

In fact, the proposed method can be applied toward the regularization of an even wider class of divergent integrals. To illustrate this concept and to demonstrate the power of the proposed methods, we consider BIE and corresponding divergent integrals that appear in the general elliptic boundary-value problem.

Let us consider a homogeneous region in which 3-D Euclidean space R^3 occupies volume V with a smooth boundary ∂V. The region V is an open bounded subset of the Euclidean space with a $C^{0,1}$ Lipschitzian regular boundary ∂V. In the region V, we consider vector functions $\mathbf{u}(\mathbf{x})$and $\mathbf{b}(\mathbf{x})$that are subject to a system of second order elliptic partial differential equations expressed in general form as

$$\mathbf{L}\cdot\mathbf{u}=\mathbf{b} \tag{2}$$

where $\mathbf{u}$and $\mathbf{b}$ are vector-functions and $\mathbf{L}$is a matrix differential operator; they have the form

$$\mathbf{L}=\begin{vmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \cdots & \vdots \\ L_{n1} & \cdots & L_{nn} \end{vmatrix}, \quad \mathbf{u}=\begin{vmatrix} u_1 \\ \vdots \\ u_n \end{vmatrix}, \quad \mathbf{f}=\begin{vmatrix} f_1 \\ \vdots \\ f_n \end{vmatrix} \tag{3}$$

The coefficients of the matrix differential operator have the form

$$L_{lk}=\frac{\partial}{\partial x_j}c_{lkji}\frac{\partial}{\partial x_i}+b_{lki}\frac{\partial}{\partial x_i}+a_{lk} \tag{4}$$

Coefficients c_{lkji}, b_{lki} and a_{lk} can be constants or can depend on coordinates.

If the region $V$is finite, it is necessary to establish boundary conditions. We consider the mixed boundary conditions in the form

$$\mathbf{u}(\mathbf{x})=\boldsymbol{\phi}(\mathbf{x}), \quad \forall\mathbf{x}\in\partial V_u, \quad \mathbf{p}(\mathbf{x})=\mathbf{P}\cdot\mathbf{u}(\mathbf{x})=\boldsymbol{\psi}(\mathbf{x}), \quad \forall\mathbf{x}\in\partial V_p \tag{5}$$

The boundary contains two parts: ∂V_u and ∂V_p such that $\partial V_u \cap \partial V_p = \varnothing$ and $\partial V_u \cup \partial V_p = \partial V$. On the part ∂V_u is prescribed an unknown function $\mathbf{u}(\mathbf{x})$, and on the part ∂V_p is prescribed its generalized normal derivative $\mathbf{p}(\mathbf{x})$. The generalized normal derivative is defined by the matrix differential operator with coefficients

$$P_{lk}=n_j c_{lkji}\frac{\partial}{\partial x_i} \tag{6}$$

here, n_i are components of the outward normal vector to the surface ∂V_p.

If the region $V$is infinite, then the solution of eq. (2) instead of the boundary conditions must satisfy additional conditions at the infinity in the form

$$\|\mathbf{u}(\mathbf{x})\|=O(r^{-1}), \quad \|\mathbf{P}\cdot\mathbf{u}(\mathbf{x})\|=O(r^{-2}) \quad \text{for} \quad r\to\infty \tag{7}$$

where $r$is the distance in the Euclidian space.

According to the generalized second Green's theorem

$$\int_V (\mathbf{u}^*\cdot\mathbf{L}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{L}^*\cdot\mathbf{u}^*)dV=\int_{\partial V}(\mathbf{u}^*\cdot\mathbf{P}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{P}^*\cdot\mathbf{u}^*)dS \tag{8}$$

we can obtain the following integral identity

$$\int_V \mathbf{u}\cdot\mathbf{L}^*\cdot\mathbf{u}^* dV=\int_{\partial V}(\mathbf{u}\cdot\mathbf{P}^*\cdot\mathbf{u}^*-\mathbf{u}^*\cdot\mathbf{P}\cdot\mathbf{u})dS-\int_V \mathbf{u}^*\cdot\mathbf{b}dV \tag{9}$$

where $\mathbf{L}^*$ is the operator adjoined to $\mathbf{L}$.

In the case of the scalar Poisson equation [5,7,39] and the system of the Lame linear equations of elasticity [1,17,40],