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Solutions of 2D and 3D non-homogeneous potential problems by using a boundary element-collocation method

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ABSTRACT

This paper presents the boundary element method for the numerical simulation of 2D and 3D nonhomogeneous potential problems. A novel technique, called recursive composite multiple reciprocity method (RCMRM), is introduced to avoid the domain integral of the non-homogenous equation in the boundary element method (BEM). The proposed method has no requirement of domain discretization, and thus is a truly boundary-type numerical method. Numerical results illustrate that the present method is computationally efficient, accurate, and convergent with an increasing number of boundary elements and collocation points.

1. Introduction

The finite element method (FEM) [1,2] and the boundary element method (BEM) [3,4] are well-established techniques applied to the solution of the engineering problems. The FEM involves a complete mesh generation of the physical domain, which is extremely costly and tedious especially for 3D unbounded or irregular domain. Unlike the FMM, the BEM only requires the boundary discretization that alleviates the computational complexity and cost.

So far, the BEM applied to the potential problem has been investigated in many branches of engineering applications [5–8]. As well known, we have to deal with a domain integral in the boundary integral equation. During the past three decades, many numerical techniques [3,9–16] have been developed to evaluate the domain integral by avoiding the discretization of the internal domain, and thus hold the advantage of the BEM in that only the boundary of the problem needs to be discretized into elements. The most popular approach is the dual reciprocity method (DRM) developed by Nardini and Brebbia [14], which has been used to solve the potential equation [17,18]. In the DRM, the non-homogeneous term is approximated with a series of radial basis functions (RBF) [19] or polynomial functions [20]. The DRM is easy programming, efficient, and flexible to handle various non-homogeneous problems. However, the distribution and location inner nodes largely affect the accuracy of the DRM. To avoid the drawback in the DRM, Gao [10] presented the radial integration method (RIM) for the evaluation of domain integrals. Based on a pure mathematical treatment, the RIM can transform domain integrals to the boundary in a unified way without using particular solutions [21–23].

On the other hand, Nowak and Brebbia [15] proposed the multiple reciprocity method (MRM) as an extension of the DRM. Compared with the DRM, the MRM has no requirement of inner point distribution. It repeatedly uses a sequence of high-order Laplace operators to transform the domain integrals to the boundary integrals. The truly boundary-type method [24,25] was developed by combining the BEM the MRM. But the standard MRM is computationally much more expensive in the construction of different interpolation matrices and has limited feasibility for general non-homogeneous problems due to its conventional use of high-order Laplacian operators.

More recently, Chen et al. [26] presented the recursive composite multiple reciprocity method (RC-MRM) which employs the high-order composite differential operators to vanish the non-homogeneous term of various types. The “recursive” technique in the RC-MRM significantly reduces CPU time and storage requirement of the original MRM. Then the boundary particle method (BPM) has been proposed by employing the strong-form boundary collocation scheme in conjunction with the RC-MRM. From now on, the BPM has been applied to Cauchy non-homogeneous Poisson and Helmholtz equations [27,28], time fractional diffusion equations [29] and Winkler plate bending problems [30]. Due to the highly ill-conditioned interpolation matrix of the strong-form collocation

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scheme, the BPM in some cases requires the regularization methods such as the truncated singular value decomposition (TSVD) to guarantee the numerical accuracy and stability. This paper makes the first attempt to introduce the RC-MRM technique into the weak-form BEM for solving 2D and 3D potential problems. Thanks to the well-posed discretization matrix of the weak-form BEM, the proposed scheme avoids the ill-conditioning of the interpolation matrix and has no requirement of regularization techniques compared with the strong-form collocation boundary schemes. The accuracy and efficiency of the present scheme are verified through several benchmark problems.

The remaining part of this paper is organized as follows. Section 2 descripts the RC-MRM technique and boundary integral equations, and then develops a new boundary-only numerical method called boundary element-collocation method. Section 3 presents the numerical implementation of the proposed method. Section 4 provides four benchmark examples to test the accuracy and efficiency of the proposed scheme. Finally, Section 5 contains some conclusions.

2. The boundary element-collocation method

The nonhomogeneous potential equation can be expressed as

$$\Delta u(\boldsymbol{x})+b(\boldsymbol{x})=0,\quad \boldsymbol{x}\in\Omega \tag{1}$$

where u is the heat potential, b is the heat source function, Ω is the domain with the boundary Γ. The following boundary conditions are imposed on Γ:

$$u(\mathbf{x})=\overline{u}(\mathbf{x}),\quad \mathbf{x}\in\Gamma_D \tag{2}$$

$$q(\boldsymbol{x})=\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}}=\overline{q}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_N \tag{3}$$

in which $\boldsymbol{n}$ is outer normal vector, and $\Gamma=\Gamma_D\cup\Gamma_N$.

2.1. Recursive composite multiple reciprocity technique

In fact, the solution of Eq. (1) can be divided into two parts: the particular solution u_p and the homogeneous solution u_h. The particular solution u_p satisfies the governing equation

$$\Delta u_p(\boldsymbol{x})+b(\boldsymbol{x})=0,\quad \boldsymbol{x}\in\Omega \tag{4}$$

And the homogeneous solution u_h needs to satisfy the following homogeneous equation and the updated boundary conditions:

$$\Delta u_h(\boldsymbol{x})=0,\quad \boldsymbol{x}\in\Omega \tag{5}$$

$$u_h(\boldsymbol{x})=\overline{u}(\boldsymbol{x})-u_p(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_D \tag{6}$$

$$q_h(\boldsymbol{x})=\overline{q}(\boldsymbol{x})-\frac{\partial u_p(\boldsymbol{x})}{\partial \boldsymbol{n}},\quad \boldsymbol{x}\in\Gamma_N \tag{7}$$

Firstly, the RC-MRM [26] technique is introduced to solve the particular solution u_p. A composite differential operator is used in the RC-MRM to vanish the non-homogeneous term b in Eq. (4) by the following iterative equation:

$$\lim_{l\to\infty} L_l\cdots L_2L_1[b(\boldsymbol{x})]=0 \tag{8}$$

in which $L_1, L_2, \cdots, L_l$ are differential operators of the same kind or different kinds. Compared with the MRM, Eq. (8) has greater flexibility and wider applicability to embrace the features of function $b(\boldsymbol{x})$, since the iterative differential operators are not restricted to the same one of the governing equation, i.e. Laplacian operators. Under the assumption that Eq. (8) is finite order or is truncated at certain order l, we have the composite MRM equation and boundary conditions as follows:

$$L_l\cdots L_2L_1\Delta u_p(\boldsymbol{x})\cong 0,\quad \boldsymbol{x}\in\Omega$$

$$\begin{aligned}&\Delta u_p(\boldsymbol{x})=-b(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma\\&L_1\Delta u_p(\boldsymbol{x})=-L_1b(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma\\&\vdots\\&L_{l-1}\cdots L_2L_1\Delta u_p(\boldsymbol{x})=-L_{l-1}\cdots L_2L_1b(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma\end{aligned} \tag{9}$$

Therefore, Eq. (4) is reduced to a higher-order homogeneous partial differential problem. A boundary collocation method, called boundary particle method (BPM) [31], can be used to solve the transformed Eq. (9) via only boundary discretization. By using this method, the solution can be approximated by a linear combination of fundamental solution as follows:

$$u_p(\boldsymbol{x})=\sum_{i=1}^{l}\sum_{j=1}^{n_p}\alpha_{ij}u_i^*(\boldsymbol{x}-\boldsymbol{s}_j) \tag{10}$$

where α_{ij} are unknown coefficients to be determined, $\boldsymbol{s}_j$ are the source points (coincide with collocation points) located on the boundary, and the function u_i^* is the fundamental solution of the composite operator L_i.

2.2. Indirect boundary integral equations (BIEs)

In the BEM literature, there are various approaches to deal with singular integrals. The virtual BEM [4,32] directly avoids singular integrations by distributing source points on a virtual boundary outside computational domain. Another approach is to use regularized techniques [3,5,33] to remove the singularity of integrals.

In this study, the homogeneous solution u_h is calculated by introducing the indirect regularized BIEs [33]

$$\int_\Gamma \psi(\boldsymbol{y})d\Gamma=0,\quad \boldsymbol{y}\in\Gamma \tag{11}$$

$$u_h(\boldsymbol{x})=\int_\Gamma \psi(\boldsymbol{y})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma+C,\quad \boldsymbol{x},\boldsymbol{y}\in\Gamma \tag{12}$$

$$\begin{aligned}q_h(\boldsymbol{x})=k\psi(\boldsymbol{x})&+\int_\Gamma[\psi(\boldsymbol{y})-\psi(\boldsymbol{x})]\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}d\Gamma\\&+\psi(\boldsymbol{x})\int_\Gamma\left[\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}+\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\right]d\Gamma,\quad \boldsymbol{x},\boldsymbol{y}\in\Gamma\end{aligned} \tag{13}$$

where $\boldsymbol{x},\boldsymbol{y}$ respectively denote the field point and source point, ψ denotes the density function to be determined, C is a unknown constant, k is 1 for interior problems or 0 for exterior problems, and

$$u^*(\boldsymbol{x},\boldsymbol{y})=\begin{cases}-\dfrac{1}{2\pi}\ln r(\boldsymbol{x},\boldsymbol{y}), & \text{for 2D}\\ \dfrac{1}{4\pi r(\boldsymbol{x},\boldsymbol{y})}, & \text{for 3D}\end{cases} \tag{14}$$

are the fundamental solutions [34] for two- and three-dimensional potential problems. A linear system of equations can be formed based on the discretization of Eq. (12) and/or Eq. (13) with N constant elements. This is a system of N equations with $N+1$ unknowns (the coefficients ψ^j $(j=1,\cdots,N)$ and C). An additional equation is obtained by discretizing Eq. (11), and then all the unknowns can be determined.

Based on Eqs. (9)–(13), we finally establish a new boundary element-collocation method for Poisson problems. In this method, α_{ij} should be firstly evaluated by using Eqs. (9) and (10), and then $u_p(\boldsymbol{x})$ at any inner or boundary point can be obtained. Then substituting $u_p(\boldsymbol{x})$ into the boundary integral Eqs. (12) and (13), we can get the unknown density function ψ. Once α_{ij},ψ have been known, the potential and its derivative at inner point can be