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# Radial integration BEM for solving non-Fourier heat conduction problems

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ABSTRACT

The radial integral BEM (RIBEM) with a step-by-step integration method is presented for solving non-Fourier heat conduction problems in this paper. First, the system of second-order ordinary differential equations is obtained by using the RIBEM to discretize the space domain. Then, the Newmark method and the central difference method are adopted to solve the system of ordinary differential equations with respect to time. Finally, several numerical examples with laser heat sources are performed to demonstrate the performance of the present method. The results show that the present approach can obtain accurate and stable numerical results.



## 1. Introduction

The unsteady heat conduction problems in general condition usually can be accurately described and solved by classical Fourier's heat conduction theory, which implies that the thermal propagation velocity is infinite. Nevertheless, classical Fourier's heat conduction theory cannot properly and accurately describe the type of problem in the situation where durations of thermal action are very short and thermal propagation velocity is finite, such as lasers and microwaves heat sources whose applications have been found in numerous different areas of material processing and medicine [1,2]. These problems need to be analyzed by the non-Fourier heat conduction theory, which takes relaxation time into account and now becomes very attractive in the research field of heat transfer.

The boundary element method (BEM) is one of the most commonly used numerical techniques with good efficiency and accuracy for solving the unsteady heat conduction problems [3]. When this the method is adopted to solve complicated unsteady heat conduction problems such as the non-Fourier heat transfer problems whose governing equation is hyperbolic partial differential equation, it is very difficult to obtain the available fundamental solutions of these problems. To overcome this disadvantage the usual approach is to make use of the fundamental solutions of approximate problems, which lead to the appearance of domain integrals in the derived integral equation. Processing of domain integrals is a key to the successful and effective implement of BEM.

The dual reciprocity method (DRM) [4] has been presented to transform the domain integrals into the boundary equivalent integrals by expanding the inhomogeneous terms into a set of global approximating basis functions whose particular solutions can be easily obtained [5,6]. The DRM has been extensively used to solve a wide range of problems including the nonlinear, non-homogeneous and non-Fourier heat conduction problems [7,8]. The dual reciprocity boundary element method (DRBEM) requires particular solutions of approximating basis functions which may be very difficult to obtain for some complicated functions. In addition, this method still requires an approximation of the known function even for known heat sources term [8].

Fortunately, in 2002, Gao [9,10] presented a very robust new transformation technique based on pure mathematical treatments, which is called the radial integration method (RIM). The RIM can transform any type of complicated domain integrals to the boundary without using particular solutions, in a simple and unified way. Moreover this method can remove various singularities appearing in domain integrals [11]. The radial integration BEM (RIBEM) [12,13], where RIM and BEM are combined, has been widely applied to many fields including the nonlinear and non-homogeneous elastic problems [14], the crack analysis in functionally graded materials [15], the viscous flow problems [16], the phase change problem [17], and the unsteady Fourier heat conduction problems [18–22]. However, there is no literature to report

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solving the non-Fourier heat conduction problems by using the RIBEM up to now.

In this paper, the RIBEM is used to solve the non-Fourier heat conduction problems. First of all, we discretize the space domain by using the RIBEM to obtain a system of second-order ordinary differential equations (ODEs) with respect to time, and then solve the ODEs by the Newmark method and the central difference method. Finally, several typical examples are presented to validate the proposed method.

## 2. Governing equations

In present study, a two-dimensional Cattaneo–Vernotte (CV) [23,24] thermal wave model is considered with constant material parameters. As a matter of fact, the governing equation of CV model, whose media is isotropic with constant thermal physical properties, is obtained by correction of the classical Fourier heat conduction equation. It holds as follows:

$$q(\mathbf{x},t+\tau)=-k\nabla^2T(\mathbf{x},t) \tag{1}$$

where $\mathbf{x}=(x_1,x_2)$, $\nabla^2=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$ is the Laplace operator, $T(\mathbf{x},t)$ is the temperature at point $\mathbf{x}\in\Omega$ and at time $t$, $k$ is the thermal conductivity, $\tau$ is the relaxation time, $q$ is heat flux. The system energy equation can be expressed as:

$$-\nabla q(\mathbf{x},t)+f(\mathbf{x},t)=\rho c\frac{\partial T(\mathbf{x},t)}{\partial t} \tag{2}$$

where $f(\mathbf{x},t)$ is a known heat source, $\rho$ is the density and $c$ is the specific heat.

After expending the left hand of Eq. (1) into Taylor series and ignoring the high order small quantity, one can obtain:

$$q(\mathbf{x},t)+\tau\frac{\partial q(\mathbf{x},t)}{\partial t}=-k\nabla T(\mathbf{x},t) \tag{3}$$

Combining Eqs. (2) and (3) and eliminating $q$, it can be expressed as:

$$\tau\rho c\frac{\partial^2T(\mathbf{x},t)}{\partial t^2}+\rho c\frac{\partial T(\mathbf{x},t)}{\partial t}=k\nabla^2T(\mathbf{x},t)+\tau\frac{\partial f(\mathbf{x},t)}{\partial t}+f(\mathbf{x},t),\quad \mathbf{x}\in\Omega \tag{4}$$

The initial condition is

$$\begin{cases} T(\mathbf{x},0)=T_0(\mathbf{x}) \\ \frac{\partial T(\mathbf{x},0)}{\partial t}=D_0(\mathbf{x}) \end{cases} \tag{5}$$

where $T_0$ and $D_0$ are prescribed functions.

The boundary conditions are

$$\begin{cases} T(\mathbf{x},t)=\overline{T}(\mathbf{x},t), & \mathbf{x}\in\Gamma_1 \\ -k\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}}=\overline{q}(\mathbf{x},t), & \mathbf{x}\in\Gamma_2 \end{cases} \tag{6}$$

where $\Gamma_1\cup\Gamma_2=\Gamma, \Gamma_1\cap\Gamma_2=\varnothing$, $\Gamma=\partial\Omega$, $\mathbf{n}$ is the unit outer normal vector, $\overline{T}$ and $\overline{q}$ are prescribed temperature history and heat flux on the boundary, respectively.

## 3. Numerical implementation of the RIBEM

### 3.1. Boundary-domain integral equation

To derive the boundary integral equation, a weight function $G$ is introduced to the governing Eq. (4) and the following domain integrals can be written as

$$\int_\Omega \tau\rho c G\frac{\partial^2T(\mathbf{x},t)}{\partial t^2}\mathrm{d}\Omega+\int_\Omega \rho c G\frac{\partial T(\mathbf{x},t)}{\partial t}\mathrm{d}\Omega = \int_\Omega kG\nabla^2T(\mathbf{x},t)\mathrm{d}\Omega+\int_\Omega \tau G\frac{\partial f(\mathbf{x},t)}{\partial t}\mathrm{d}\Omega+\int_\Omega Gf(\mathbf{x},t)\mathrm{d}\Omega \tag{7}$$

The function $G$ is the fundamental solution of potential problems, and it can be expressed as

$$G(\mathbf{x},\mathbf{y})=\frac{1}{2\pi}\ln\frac{1}{r(\mathbf{x},\mathbf{y})} \tag{8}$$

where $r(\mathbf{x},\mathbf{y})$ is the distance between the source point $\mathbf{y}$ and the field point $\mathbf{x}$.

Using Gauss' divergence theorem and the property of Dirac delta function, the first domain integral at the right hand side can be manipulated as

$$\int_\Omega kG(\mathbf{x},\mathbf{y})\nabla^2T(\mathbf{x},t)d\Omega = \int_\Gamma\left(-G(\mathbf{x},\mathbf{y})q-kT(\mathbf{x},t)\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}}\right)d\Gamma-kT(\mathbf{y},t) \tag{9}$$

where $q=-k\,\partial T/\partial\mathbf{n}$.

Substituting Eq. (9) into Eq. (7), it follows that

$$kT(\mathbf{y},t)=\int_\Gamma\left(-Gq-kT\frac{\partial G}{\partial \mathbf{n}}\right)\mathrm{d}\Gamma+\int_\Omega\left(\tau G\frac{\partial f}{\partial t}+Gf\right)\mathrm{d}\Omega -\rho c\int_\Omega G\frac{\partial T}{\partial t}\mathrm{d}\Omega-\tau\rho c\int_\Omega G\frac{\partial^2 T}{\partial t^2}\mathrm{d}\Omega \tag{10}$$

Eq. (10), where boundary integrals and domain integrals are both involved, is valid only for internal points. For boundary points, a similar integral equation can be obtained by letting $\mathbf{y}\to\Gamma$ as is done in the conventional BEM [25]. A general integral equation is presented as following:

$$c(\mathbf{y})kT(\mathbf{y},t)=\int_\Gamma\left(-Gq-kT\frac{\partial G}{\partial \mathbf{n}}\right)\mathrm{d}\Gamma+\int_\Omega\left(\tau G\frac{\partial f}{\partial t}+Gf\right)\mathrm{d}\Omega -\rho c\int_\Omega G\frac{\partial T}{\partial t}\mathrm{d}\Omega-\tau\rho c\int_\Omega G\frac{\partial^2 T}{\partial t^2}\mathrm{d}\Omega \tag{11}$$

where

$$c(\mathbf{y})=\begin{cases} 1, & \mathbf{y}\in\Omega \\ \frac{\varphi(\mathbf{y})}{2\pi}, & \mathbf{y}\in\Gamma \end{cases} \tag{12}$$

in which $\varphi(\mathbf{y})$ is the interior angle at a point $\mathbf{y}$ of the boundary $\Gamma$. Particularly, $c(\mathbf{y})=0.5$ if $\mathbf{y}$ is a smooth point on the boundary.

### 3.2. Transformation of domain integrals to the boundary by RIM

The three domain integrals involved in Eq. (11) need to be transformed into equivalent boundary integrals by RIM [9–11]. In order to describe the transformation process clearly, we assume that the boundary $\Gamma$ is discretized into $N_b$ linear elements, $N_I$ internal nodes are distributed in the domain $\Omega$ and the total number of nodes is $N=N_b+N_I$. And numbering of nodes firstly begins from the boundary nodes in this paper.

#### 3.2.1. Transformation of the first domain integral

In general, the heat source term $f(\mathbf{x},t)$ is a known function. Therefore, the RIM can be used directly to transform the first domain integral in Eq. (11) into the boundary as follows:

$$\int_\Omega\left(\tau G\frac{\partial f}{\partial t}+Gf\right)\mathrm{d}\Omega=\int_\Gamma\frac{1}{r(\mathbf{z},\mathbf{y})}\frac{\partial r}{\partial n}F^A(\mathbf{z},\mathbf{y},t)d\Gamma(\mathbf{z}) \tag{13}$$

where the radial integral $F^A$ can be expressed as

$$F^A(\mathbf{z},\mathbf{y},t)=\int_0^{r(\mathbf{z},\mathbf{y})}\left[\tau G(\mathbf{x},\mathbf{y})\frac{\partial f(\mathbf{x},t)}{\partial t}+G(\mathbf{x},\mathbf{y})f(\mathbf{x},t)\right]\xi d\xi \tag{14}$$

In Eqs. (13) and (14), it is noted that $\mathbf{z}$ is the boundary point and the variable transformation relationship about $\mathbf{x}$ can be given by [9]

$$\mathbf{x}=\mathbf{y}+\hat{\mathbf{r}}\xi \tag{15}$$

In Eq. (15), $\hat{\mathbf{r}}=(\mathbf{z}-\mathbf{y})/r(\mathbf{z},\mathbf{y})$ is a unit vector. Eq. (12) can also be expressed as a component form, i.e., $x_i=y_i+r_{,i}\xi$ where