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Calculation of three-dimensional nearly singular boundary element integrals for steady-state heat conduction

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ABSTRACT

In this work, a novel approach is presented for three-dimensional nearly singular boundary element integrals for steady-state heat conduction. Accurate evaluation of the nearly singular integrals is an important issue in the implementation of boundary element method (BEM). In this paper, an exponential transformation is introduced to deal with the nearly singular integrals in three-dimensional BEM. First, a triangle polar coordinate system is introduced. Then, the exponential transformation is performed by five steps. For each step, a new transformation is proposed based on the distance from the source point to surface elements which is expressed as $r^2 = O(A_k^2(\theta)\rho^2 + r_0^2)$, and all steps can finally be unified into a uniform formation. Moreover, to perform integrations on irregular elements, an adaptive integration scheme considering both the element shape and the projection point associated with the proposed transformation is introduced. Numerical examples are presented to verify the proposed method. Results demonstrate the accuracy and efficiency of our method.

1. Introduction

Dealing with singular integrals and nearly singular integrals has been a seemingly daunting task since the early days of the boundary element method (BEM) [1–6]. For singular integrals, many numerical techniques have been proposed, such as transformation techniques [7], regularization techniques [6,8,9], radical integration techniques [10,11] and Taylor series expansion techniques [12,13]. Thus in this paper, we focus on the nearly singular integrals.

Near singularities are involved in many boundary element method (BEM) analyses of engineering problems, such as problems on thin shell-like structures [5,14,15], the contact problems [16], as well as the sensitivity problems [17]. It can be found in the pioneer work of Liu [18–20]. Accurate and efficient evaluation of nearly singular integrals with various kernel functions of the type $O(1/r^{\alpha})$ is crucial for successful implementation of the boundary type numerical methods based on boundary integral equations (BIEs), such as the boundary element method (BEM), the boundary face method (BFM) [7,21]. A near singularity arises when a source point is close to but not on the integration elements. Although these integrals are really regular in nature, they can't be evaluated accurately by the standard Gaussian quadrature. This is because the denominator r, the distance between the source and the field point, is close to zero but not zero. The difficulty encountered in the numerical evaluation mainly results from the fact that the integrands of nearly singular integrals vary drastically with respect to the distance. Effective computation of nearly singular integrals has received intensive attention in recent years. Various numerical techniques have been developed to remove the near singularities, such as nonlinear transformations [19], Taylor series expansion algorithm [22], global regularization [6,23], optimization transformation [24], semi-analytical or analytical integral formulas [25–29], the sinh transformation [30–34], polynomial transformation [35], adaptive subdivision method [7,36], distance transformation technique [21,38–40], the PART method [41–43], and the exponential transformation [44–46]. Most of them benefit from the strategies for computing singular integrals [6,22,23].

Among those techniques, the exponential transformation technique seems to be a more promising method for dealing with different orders of nearly singular boundary element integrals. However, the transformation is only limited to 2D boundary element. In this paper, we develop the exponential transformation

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technique for steady-state heat conduction to 3D boundary element method in triangle polar coordinate system.

In our method, first a triangle polar coordinate system introduced. This system is very similar to the polar system, but its implementation is simpler than the polar system and also performs efficiently. Then by applying the first order Taylor series expansion, the distance between the source point and the integration elements is the equivalent of $A_k^2(\theta)\rho^2+r_0^2$. Based on the equivalent distance, the exponential transformation in Refs. [44–46] can be developed to 3D BEM in the triangle polar coordinate system. Using the proposed transformation, the integrals with near singularities can be accurately calculated. Furthermore, an element subdivision technique considering both the element shape and the positions of the project point is introduced in combination with the improved transformation to perform integrations on irregular elements. With our method, the nearly singular boundary element integrals of regular or irregular elements can be accurately and effectively calculated. Numerical examples are presented to validate the proposed method. Results demonstrate the accuracy and efficiency of our method.

This paper is organized as follows. The general form of nearly singular integrals is described in Section 2. In Section 3, distance function is constructed in triangle polar coordinate system. In Section 4, the transformations for nearly singular integrals are presented and the element subdivision technique is introduced. Numerical examples are given in Section 5. The paper ends with conclusions in Section 6.

2. General descriptions

In this section, we will give a general form of the nearly singular integrals over 3D boundary elements. First we consider the boundary integral equations for 3D steady-state heat conduction.

The well-known BIE for steady-state heat conduction in 3-D is

$$C(y)u(y)=\int_\Gamma u(\mathbf{x})q^*(\mathbf{x},\mathbf{y})d\Gamma-\int_\Gamma q(\mathbf{x})u^*(\mathbf{x},\mathbf{y})d\Gamma \tag{1a}$$

$$q_i(y)=\int_\Gamma q(\mathbf{x})\frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial y_i}d\Gamma-\int_\Gamma u(\mathbf{x})\frac{\partial q^*(\mathbf{x},\mathbf{y})}{\partial y_i}d\Gamma, i=1,2,3 \tag{1b}$$

where $\mathbf{x}$ and $\mathbf{y}$ represent the field point and the source point in the BEM, with components x_i and y_i, $i=1,2,3$, respectively. Eq. (1b) is used for calculation of the flux components at the domain points.

Eqs. (1a) and (1b) is discretized on the boundary Γ by boundary elements $\Gamma_e(e=1\sim N)$ defined by interpolation functions. The integral kernels of Eqs. (1a) and (1b) become nearly singular when the distance between the source point and integration element is very small compared to the size of integration element. And the integrals in Eqs. (1a) and (1b) become nearly singular with different orders, namely, $u^*(\mathbf{s},\mathbf{y})$ with near weak singularity, $q*(\mathbf{s},\mathbf{y})$ and $\partial u^*(\mathbf{x},\mathbf{y})/\partial y_i$ with near strong singularity, $\partial q^*(\mathbf{x},\mathbf{y})/\partial y_i$ with near hyper-singularity.

In this paper, we develop the exponential transformation method for various boundary integrals with near singularities of different orders. The new method is detailed in following sections. For the sake of clarity and brevity, we take following integrals as a general form to discuss:

$$I=\int_S\frac{f(\mathbf{x},\mathbf{y})}{r^\alpha}dS,\quad \alpha=1,3,5\quad r=||\mathbf{x}-\mathbf{y}||_2 \tag{2}$$

where f is a smooth function, $\mathbf{x}$ and $\mathbf{y}$ represent the field point and the source point in BEM, with components x_i and y_i, respectively. S represents the boundary element. We assume that the source point is close to S, but not on it.

3. Construction distance function in the triangle polar coordinate system

3.1. The triangle polar coordinate system

As shown in Fig. 1, a plane triangle is mapped onto a square of unit side-length. Within the framework of the BEM, the triangle represents a boundary element embedded in a local coordinate system. The following mapping [7,21] scheme is used:

$$\begin{cases}\xi=\xi_0+(\xi_1-\xi_0)\rho_1+(\xi_2-\xi_1)\rho_1\rho_2\\ \eta=\eta_0+(\eta_1-\eta_0)\rho_1+(\eta_2-\eta_1)\rho_1\rho_2\end{cases}\quad \rho_1,\rho_2\in[0,1] \tag{3}$$

The Jacobian for the transformation from (ξ,η) system to system triangle polar coordinate system is $\rho_1 S_\Delta$, and

$$S_\Delta=|\xi_1\eta_2+\xi_2\eta_0+\xi_0\xi_2-\xi_2\eta_1-\xi_0\eta_2-\xi_1\eta_0| \tag{4}$$

It should be noted that triangle polar coordinate system is analogous to the polar system, because the performances of ρ_1 and ρ_2 are similar to the ρ and θ, respectively. However, this new system is much simpler and even more effective than the polar system. This is due to the fact that both ρ_1 and ρ_2 are constrained to the interval [0,1] in each triangle. So the complicated determination of ρ and θ in each triangle is avoided.

3.2. The definition of the distance

As shown in Fig. 2, employing the first-order Taylor series expansion in the neighborhood of the projection point $\mathbf{x}^c$, we have

$$x_k-y_k=x_k-x_k^c+x_k^c-y_k=\frac{\partial x_k}{\partial\xi}|_{\xi_1=\xi_0}$$
$$\eta_1=\eta_0(\xi-\xi_0)+\frac{\partial x_k}{\partial\eta}|_{\xi_1=\xi_0}$$
$$\eta_1=\eta_0(\eta-\eta_0)+r_0n_k(\xi_0,\eta_0)+O(\rho_1^2)=\rho_1A_k(\beta)+r_0n_k(\xi_0,\eta_0)+O(\rho_1^2) \tag{5}$$

where

$$A_k(\beta)=\left(\frac{\partial x_k}{\partial\xi}\right)|_{\substack{\xi_1=\xi_0\\ \eta_1=\eta_0}}[(\xi_1-\xi_0)+(\xi_2-\xi_1)\rho_2]+\left(\frac{\partial x_k}{\partial\eta}\right)|_{\substack{\xi_1=\xi_0\\ \eta_1=\eta_0}}[(\eta_1-\eta_0)+(\eta_2-\eta_1)\rho_2]$$

Using Eqs. (3)–(5), we can easily obtain the distance function in the following form:

$$r^2=(x_k-y_k)(x_k-y_k)=A_k^2(\rho_2)\rho_1^2+r_0^2+O(\rho_1^3) \tag{6}$$

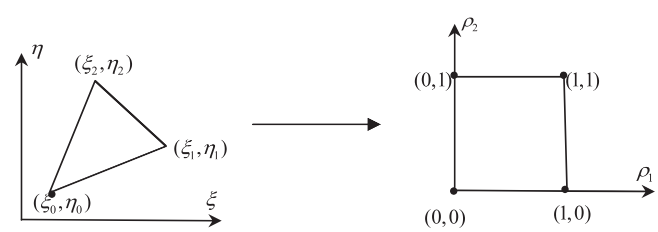

Fig. 1. The triangle polar coordinate system.

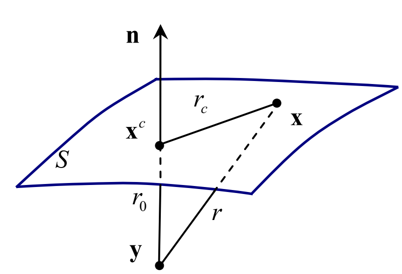

Fig. 2. Minimum distance r_0, from the source point $\mathbf{y}$ to curved surface element.