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A comparative study of meshless complex quadrature rules for highly oscillatory integrals

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ABSTRACT

In this paper a stable and modified form of the Levin method based on Bessel radial basis functions is employed for numerical solution of highly oscillatory integrals. In the proposed technique, the multiquadric radial basis function (Levin, 1982 [1]; Siraj-ul-Islam et al., 2013 [2]) is replaced by Bessel radial basis functions (Fornberg et al., 2006 [3]) and thin plate spline of order three. In this scheme the integration form is first transformed into differential form and then the numerical solution of the corresponding differential form is found. The accuracy and the algebraic stability in the form of well-conditioned coefficient matrices of the proposed methods are confirmed through numerical experiments.

1. Introduction

The exact solution of highly oscillatory integrals and integrals with and without stationary points is difficult to obtain. So, an alternate way is to use numerical techniques, for highly oscillatory integrals. Computational scientists and mathematicians have focused their attention on the construction of efficient and accurate numerical methods in an effort to encounter challenges being faced in the evaluation of highly oscillatory integrals. Highly oscillatory integrals occur in a variety of applications including quantum mechanics, electromagnetic waves, optics and acoustics and their correct evaluation is one of the key research problems [4–10].

Strategies regarding finding numerical solution of highly oscillatory integrals reported in the literature are sparse and this area has remained unattended by and large. In the absence of specialized and accurate procedures, conventional quadrature methods were mostly used for the numerical solutions of highly oscillatory integrals. The use of conventional quadrature rules is counter productive due to bad accuracy and high computational cost.

In a univariate case, the above-mentioned highly oscillatory integrals are uniformly represented as

$$I = \int_a^b f(x) e^{iwg(x)}\, dx \qquad (1)$$

where $f(x)$ is called the amplitude, $g(x)$ is a phase function, $f(x)$ and $g(x)$ are both smooth functions. The parameter $\omega \geq 0$, called the frequency parameter, mainly responsible for oscillatory behavior of the integrand. The integrand given in Eq. (1) may or may not have critical point(s). In the case of stationary points of Eq. (1) at a point x, $g'(x) = 0$ for $x \in [a, b]$.

When there are no stationary points in $[a, b]$ and larger the frequency ω, the traditional methods like Gauss–Legendre quadrature, Simpson rule, etc. fail to approximate the integral given in Eq. (1) accurately and efficiently. In the case of highly oscillatory integrals, apart from accuracy, these methods do not converge fast to the actual solution even on a dense grid. Therefore, specialized algorithms are needed to get an accurate and a stable solution of such type of integrals.

The methods specified for the numerical solution of highly oscillatory integrals are categorized into two main groups, Levin's type of method [1] and Filon type of method [11]. The former converts the oscillatory integrals into a differential form and subsequently collocate monomial [1] basis or any other basis like multiquadric [2] to get solution of the corresponding differential equation. Filon's approach tackles such integrals in a different way which is mainly based on the asymptotic theory. Some of the methods which are based on Filon's approach and are specially designed for numerical solution of highly oscillatory integrals are reported in [11–16]. The contributions based on Levin's approach are reported in [1,2,4,5,17,18]. The limitation of the Filon method is that they can only deal with linear phase functions, whereas in

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practical situation one may have nonlinear phase functions as well. The specific contributions of the paper are the following:

(i) Numerical approximation of highly oscillatory integrals using modified Levin's method based on meshless procedure using Bessel RBFs of orders 5, 7 and 9 and thin plate spline of order three.
(ii) The method is validated on different nonlinear, linear and periodic oscillators.
(iii) The comparative performance of different basis functions namely MQ RBF, BRBF5–BRBF9 and TPS3, in the context of numerical solution of highly oscillatory integrals with and without stationary point has been thoroughly examined.

To cope with issues related to highly oscillatory integrals, a hybrid method based on MQ RBF and hybrid functions has been presented in [2]. The method [2] which is based on MQ RBF is the modified form of the traditional Levin's approach [1], where monomial basis has been replaced by MQ RBF. Due to spectral accuracy, MQ RBF is a preferred choice versus monomial. The applications of the MQ RBF, which are related to the numerical solution of the PDEs, are included in [7,18–26].

Apart from accuracy, both the algorithms [1,2] are still confronted with a major issue of ill-conditioned coefficient matrices. The resulting linear systems are often ill-conditioned, which is the main reason behind the numerical instability. In the case of monomials, it is the Vandermonde matrix which is highly ill-conditioned and in the case of MQ RBF, it is dense RBF matrix depending on a shape parameter c which causes ill-conditioning. The choice of shape parameter c in the MQ RBF can destabilize the algorithm if chosen out side a permissible range. One cannot find at the same time both, good conditioning and good accuracy, in numerical methods [1,2].

In this situation a relatively better trade-off between accuracy and numerical stability of the algorithm [2] is proposed in this paper. In the current work, we suggest a meshless procedure based on truncated oscillatory radial basis functions [3] instead of monomials and MQ RBF. Truncated oscillatory radial basis functions have been investigated in [3], and can be written as

$$\phi(r)=\frac{J_{(d/2)-1}(cr)}{(cr)^{(d/2)-1}},\quad d=1,2,3,\ldots \tag{2}$$

where J_α represents a Bessel function of order α, c is the shape parameter and r is the radial distance.

Truncated oscillatory radial basis functions [3] have a dual advantage in terms of less shape parameter sensitivity and a well-conditioned linear system. These RBFs have some disadvantages in terms of accuracy which is lower than the MQ RBF method [2] and the fact that we have not found any apparent relationship between the inherent oscillatory character of the Bessel radial basis functions for the solution of oscillatory integrals. Due to advantages mentioned earlier, Bessel radial basis functions are worth investigations in the context of different numerical applications. Truncated oscillatory radial basis functions are using Bessel functions as building blocks and henceforth named as Bessel radial basis functions. In this paper we consider the proposed method based on Bessel radial basis functions for univariate highly oscillatory integrals. Extension of the proposed algorithm to multi-variate cases is a straight forward procedure.

The paper is organised as follows: in Section 2, the meshless procedure based on different types of RBFs is described. In Section 3, numerical results and discussion are given. In Section 4, some conclusion are given.

2. Meshless procedure

A univariate continuous function, defined for r which is a positive real number, may or may not have a free parameter called an RBF $\phi(r)$. The free parameter is called the shape parameter of the RBF, where c stands for the shape parameter. The following form is used for an RBF interpolant, for a set of n centers $x_1^c,\ldots,x_n^c$, given in R:

$$P(x)=\sum_{k=1}^{n}\beta_k\phi(\|\mathbf{x}-x_k^c\|_2,c),\quad \mathbf{x}\in R. \tag{3}$$

where β_k, $k=1,2,\ldots,n$ are the RBFs coefficients.

In this paper we propose Bessel radial basis functions of orders 5, 7 and 9 and thin plate spline of order 3 instead of monomials and MQ RBF for the numerical solution of the differential form resulted to Levin's method.

The Bessel radial basis functions of orders 5, 7 and 9 are described in the following order [3].

Bessel radial basis function of order 5 (BRBF5) is given as follows:

$$\phi(r)=\sqrt{\frac{2}{\pi}}\left[\frac{\sin(cr)-cr\cos(cr)}{(cr)^3}\right]. \tag{4}$$

Bessel radial basis function of order 7 (BRBF7) can be written as

$$\phi(r)=\sqrt{\frac{2}{\pi}}\left[\frac{3\sin(cr)-3\,cr\cos(cr)-(cr)^2\sin(cr)}{(cr)^5}\right]. \tag{5}$$

Bessel radial basis function of order 9 (BRBF9) is given as follows:

$$\phi(r)=\sqrt{\frac{2}{\pi}}\left[\frac{15\sin(cr)-15\,cr\cos(cr)-6\,(cr)^2\sin(cr)+(cr)^3\cos(cr)}{(cr)^7}\right]. \tag{6}$$

where $r=(x-x_i)$, $i=1,2,\ldots,n$.

In [3], the following theorem which provides a foundation for the existence of inversion of the coefficient matrix for the interpolation problems has been proved.

Theorem 1. *The radial functions given by Eq. (2) will give non-singular interpolation in up to d dimensions when* $d\geq 2$.

Similarly thin plate spline of order 3 (TPS3) which is a shape parameter free RBF is defined as

$$\phi(r)=(\sqrt{r^2})^3, \tag{7}$$

where $r^2=(x-x_i)^2$, $i=1,2,\ldots,n$.

MQ radial basis function which has been used in [2] for highly oscillatory integrals is given as follows:

$$\phi(r)=\sqrt{c^2+r^2} \tag{8}$$

where $r^2=(x-x_i)^2$, $i=1,2,\ldots,n$ and c is the shape parameter.

The first derivative of BRBF5 is given as

$$\phi_{(5d)}=\frac{d}{dr}(\phi(r))=\sqrt{\frac{2}{\pi}}c\left[\frac{\sin(cr)}{(cr)^2}-\frac{3(\sin(cr)-cr\cos(cr))}{(cr)^4}\right]. \tag{9}$$

The first derivative of BRBF7 is given as

$$\phi_{(7d)}=\frac{d}{dr}(\phi(r))=\sqrt{\frac{2}{\pi}}c\left[\frac{(6cr\sin(cr))-(cr)^2\cos(cr)+(15\cos(cr))}{(cr)^5}-\frac{(15\sin(cr))}{(cr)^6}\right]. \tag{10}$$

The first derivative of BRBF9 is given as

$$\phi_{(9d)}=\frac{d}{dr}(\phi(r))=\sqrt{\frac{2}{\pi}}c\left[\frac{45\sin(cr)-(10cr\cos(cr))-(cr)^2\sin(cr)}{(cr)^6}+\frac{(105\cos(cr))}{(cr)^7}-\frac{(105\sin(cr))}{(cr)^8}\right]. \tag{11}$$