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An image denoising approach based on a meshfree method and the domain decomposition technique

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ABSTRACT

In this paper the meshfree finite point method (FPM) with domain decomposition is investigated for solving a nonlinear PDE to denoise digital images. The obtained algorithm is parallel and ideal for parallel computers. We use the scheme of Catté et al. [9] and we believe that this method could be successfully implemented for other noise removal schemes. The finite point method is a meshfree method based on the point collocation of moving least squares approximation. This method is easily applicable to nonlinear problems due to the lack of dependence on a mesh or integration procedure. Also computer experiments indicate the efficiency of the proposed method.

1. Introduction

The Perona–Malik equation [43], proposed in 1990, has stimulated a great deal of attention in image processing among the denoising techniques based on anisotropic diffusion equations. It is commonly believed that Perona–Malik equation provides a potential algorithm for noise removing, image segmentation, edge detection and image enhancement [25]. The basic idea of Perona–Malik algorithm is to evolve an initial image, $u^0(\mathbf{x})$, defined in a domain $\Omega \subseteq \mathbb{R}^n$ $(n=2,3)$, under a diffusion operator with the edge controlling property [43]

$$u_t - \nabla \cdot (g(|\nabla u|)\nabla u) = 0, \tag{1.1}$$

where $u(t,\mathbf{x})$ is an unknown function defined in $I \times \Omega$. The equation is accompanied by zero Neumann boundary conditions and the initial condition

$$\frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } I \times \partial\Omega,$$
$$u(0,\mathbf{x}) = u^0(\mathbf{x}) \quad \text{in } \Omega, \tag{1.2}$$

where n is the unit outward normal to the boundary of Ω.

We note that if $g(s)$ is decreasing, the Perona–Malik equation can behave locally like the backward heat equation, which is an ill-posed problem. So, for $g(s)$ used in practice both the existence and uniqueness of a solution cannot be obtained [35]. In order to overcome the mathematical disadvantage and inhibit the influence of the noise, Catté et al. [9] proposed the following version of the edge indicator:

$$u_t - \nabla \cdot (g(|\nabla G_\sigma * u|)\nabla u) = f(u^0 - u) \quad \text{in } \Omega \times I, \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded rectangular domain, $I=[0,T]$ is a scaling interval and function f is the Lipschitz continuous, nondecreasing with $f(0)=0$. The diffusion coefficient $g : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a nonincreasing function on the magnitude of local image gradient ∇u and has such properties: $g(\sqrt{s})$ is smooth, $g(0)=1$, and we admit

$$\lim_{s\to\infty} g(s) = 0.$$

Two commonly used diffusion coefficients are

$$g(s) = \exp(-(s/K)^2), \tag{1.4}$$

and

$$g(s) = \frac{1}{1+(s/K)^2}. \tag{1.5}$$

$G_\sigma \in C^\infty(\mathbb{R}^2)$ is a smoothing kernel with

$$\int_{\mathbb{R}^2} G_\sigma(\mathbf{x})\, d\mathbf{x} = 1, \quad \int_{\mathbb{R}^2} |\nabla G_\sigma|\, d\mathbf{x} \le C_\sigma.$$

Moreover

$$\lim_{\sigma\to 0} G_\sigma = \delta_{\mathbf{x}},$$

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where $\delta_{\mathbf{x}}$ is the Dirac measure at point $\mathbf{x}$. Also

$$G_\sigma * u = \int_{\mathbb{R}^2} G_\sigma(\mathbf{x}-\eta)\tilde{u}(\eta)\, d\eta,$$

where $\tilde{u}$ is an extension of u to $\mathbb{R}^2$ [9]. The initial condition $u^0(x)$ represents a gray-level intensity function of the initial processed image [29]. For our presentation we have chosen this well known nonlinear diffusion model.

Researches around image processing have been regarded toward these aspects: studying the mathematical properties of the nonlinear diffusion and the related variational formulation [9,58,8], introducing novel models and analyzing the well-posedness and the stability of the introduced models [1,2,20,28,45], improving and modifying the anisotropic diffusion [10,25,26,30,34], and studying the relations between the anisotropic diffusion methods and other image processing methods [6], etc.

Among these researches, different numerical methods have been used for solving the proposed models. For example the finite difference method [26] and the domain decomposition technique in finite difference method [22] are discussed to solve nonlinear problems in image denoising. Also the well-established variational computational techniques, namely, finite element, finite volume and complementary volume methods, to solve nonlinear problems in image multiscale analysis are discussed in [27,29,36,35], etc. Our approach in the current paper is different. In this work we present the meshfree finite point method for the nonlinear diffusion equation arising in image denoising.

Recently some attentions have been paid to the meshfree methods, particularly moving least squares (MLS) based methods, for the numerical solution of partial differential equations. Using this approximation some well known methods such as element free Galerkin (EFG) method [7,19], boundary node method (BNM) [39], meshless local boundary integral equation (LBIE) method [12,50,51], meshless local Petrov–Galerkin (MLPG) method [3–5,13,37], finite point method [40–42,55,56] and other relative methods [14,46] have been constructed.

The finite point method (FPM) proposed in [40] is implemented by collocating the moving least squares approximation around each point in the governing partial differential equations. Meshfree collocation methods (or meshfree strong-form methods) have a long history. To approximate strong-form of PDEs using meshfree methods, the PDE is usually discretized at nodes by some forms of collocation. As mentioned in [31], meshfree strong-form methods have the following advantages:

- The procedure of discretizing the governing equations is straightforward and the algorithms for implementing the discretized equations are simple.
- They are, in general, computationally efficient. Due to discretizing the PDEs directly without using weak-forms, no numerical integration is required.
- They are truly meshless, i.e. no mesh is used for both approximations and numerical integrations.

For more details about meshfree methods see [3,15–18,31,32,38,47,49,52].

The organization of this paper is as follows: in Section 2 the mathematical formulation, including the discretization of the nonlinear diffusion model, the overlapping domain decomposition method and the finite point method are discussed. The numerical results and the computational aspects to test the accuracy and efficiency of the proposed method are presented in Section 3. In order to have a comparison, the results of meshfree RBF method are presented in this section too. Finally, some concluding remarks are given in Section 4.

2. Mathematical formulation

2.1. The time discretization

In this section we study the discretization in time of Eq. (1.3). Similar to the approach in [36] we discretize the scaling interval $[0, T]$ and replace the scale derivative in (1.3) by a backward difference operator. The nonlinear terms of the equations are treated from the previous time step while the linear terms are considered on the current time level. This means semi-implicitness of the method. So the approximation in scale of Eq. (1.3) can be considered as

$$\frac{u^k-u^{k-1}}{\tau} - \nabla\cdot(g(|\nabla G_\sigma * u^{k-1}|)\nabla u^k) = f(u^0-u^{k-1}), \tag{2.1}$$

in which $\tau=\Delta t$. Before going to the space discretization we focus on the realization of the convolution included in the evaluation of function g in (2.1). To deal with the convolution term we follow the interesting strategy used in [29]. Considering the Gaussian function, i.e.

$$G_\sigma = \frac{1}{(4\pi\sigma)^{N/2}}\exp\left(\frac{-|x|^2}{4\sigma}\right),$$

as the smoothing kernel G_σ, the term $G_\sigma * u^{k-1}$ could be replaced by solving the heat equation for time σ with the initial condition given by u^{k-1}. This linear equation can be solved numerically at the same domain by just one implicit step with length σ. Thus instead of computing $G_\sigma * u^{k-1}$ directly, we look for the solution u^c of the heat equation discretized in time by the backward Euler method with step σ:

$$\frac{u^c-u^{k-1}}{\sigma} = \Delta u^c, \tag{2.2}$$

where Δ denotes the Laplace operator. Due to the strong form in the finite point method and refusing of mesh generation in order to compute the numerical integrations, this strategy is very efficient and suitable to that of computing $G_\sigma * u^{k-1}$ directly.

So Eq. (2.1) can be simplified as

$$\frac{u^k-u^{k-1}}{\tau} - \nabla\cdot(g(|\nabla u^c|)\nabla u^k) = f(u^0-u^{k-1}), \tag{2.3}$$

in which u^c is the solution of Eq. (2.2) for each time step. The resultant equation at each time step can be rewritten as

$$\mathcal{L}u^k = \mathcal{G}(u^{k-1}) \quad \text{in } \Omega,$$
$$\frac{\partial u^k}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

in which

$$\mathcal{L}u^k = u^k - \tau\left(g(|\nabla u^c|)\Delta u^k + \frac{\partial g(|\nabla u^c|)}{\partial x}\frac{\partial u^k}{\partial x} + \frac{\partial g(|\nabla u^c|)}{\partial y}\frac{\partial u^k}{\partial y}\right), \tag{2.5}$$

and

$$\mathcal{G}(u^{k-1}) = \tau f(u^0-u^{k-1}) + u^{k-1}. \tag{2.6}$$

2.2. The space discretization

2.2.1. The overlapping domain decomposition method

The most known domain decomposition method was introduced by Schwarz in 1870. Not originally intended as a numerical method, the classical alternating Schwarz method may be used to solve elliptic boundary value problems on domains that are the union of two subdomains. In the sense that solving the original problem can be derived by alternatingly solving the same elliptic boundary problem restricted to the individual subdomains [53]. Domain decomposition methods have become essential tools in