Engineering Analysis with Boundary Elements 39 (2014) 121–128

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

# Comparison of meshless local weak and strong forms based on particular solutions for a non-classical 2-D diffusion model

Saeid Abbasbandy [a,*], Hadi Roohani Ghehsareh [b], Mohammed S. Alhuthali [c], Hamed H. Alsulami [c]

[a] Department of Mathematics, Imam Khomeini International University, Ghazvin 34149-16818, Iran
[b] Department of Mathematics, Malek Ashtar University of Technology, Shahin Shahr, Isfahan 83145-115, Iran
[c] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

ARTICLE INFO

*Article history:*
Received 30 April 2013
Accepted 1 November 2013
Available online 25 December 2013

*Keywords:*
MLPG method
Non-local boundary value problem
The method of approximate particular solutions

ABSTRACT

In the current work, a new aspect of the weak form meshless local Petrov–Galerkin method (MLPG), which is based on the particular solution is presented and well-used to numerical investigation of the two-dimensional diffusion equation with non-classical boundary condition. Two-dimensional diffusion equation with non-classical boundary condition is a challenged and complicated model in science and engineering. Also the method of approximate particular solutions (MAPS), which is based on the strong formulation is employed and performed to deal with the given non-classical problem. In both techniques an efficient technique based on the Tikhonov regularization technique with GCV function method is employed to solve the resulting ill-conditioned linear system. The obtained numerical results are presented and compared together through the tables and figures to demonstrate the validity and efficiency of the presented methods. Moreover the accuracy of the results is compared with the results reported in the literature.



## 1. Introduction

Many of natural Phenomena in Science and Engineering have been modelled by non-classical boundary value problems. In these non-classical models often some integral terms appear in the boundary conditions or in the governing equations. These types of problems constitute a special class of boundary value problems which are widely appeared for mathematical modelling of various processes of physics, heat transfer, ecology, thermoelasticity, chemistry, biology, and industry [1–6]. In the current work a numerical investigation would be given to approximate the solution of the following two-dimensional diffusion equation:

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\quad \mathbf{x}=(x,y)\in\Omega\subseteq\mathbb{R}^2,\ \ t\geq 0, \tag{1.1}$$

with the following initial and boundary conditions:

$$\begin{aligned}
&u(\mathbf{x},0)=f(\mathbf{x}),\quad \mathbf{x}\in\Omega,\\
&u(\mathbf{x},t)=h_1(\mathbf{x},t),\quad \mathbf{x}\in\Gamma_1,\ \ t\geq 0,\\
&u(\mathbf{x},t)=k(\mathbf{x})\mu(t),\quad \mathbf{x}\in\Gamma_2,\ \ t\geq 0,\\
&\frac{\partial u}{\partial n}(\mathbf{x},t)=h_2(\mathbf{x},t),\quad \mathbf{x}\in\Gamma_3,\ \ t\geq 0,
\end{aligned} \tag{1.2}$$

and the non-classical boundary condition:

$$\iint_\Omega u(\mathbf{x},t)\,d\mathbf{x}=m(t),\quad t\geq 0, \tag{1.3}$$

where $u(\mathbf{x},t)$ and $\mu(t)$ are unknown functions would be determined, the positive constant $\alpha$ denotes the thermal diffusivity, $f(\mathbf{x}), h_1(\mathbf{x},t), h_2(\mathbf{x},t)$ and $k(\mathbf{x})$ are given sufficiently smooth functions, $\Gamma=(\Gamma_1\cup\Gamma_2\cup\Gamma_3)$ is the closed curve bounding the region $\Omega$, ($\Gamma_1,\Gamma_2,\Gamma_3$ are non-intersecting curves), $\overline{\Omega}=(\Omega\cup\Gamma)\in\mathbb{R}^2$ denotes the spatial domain and $t$ is time.

The presence of the nonclassical term (1.3) in the boundary conditions causes that the theoretical study of the problem is connected with great difficulties and also the implementation of many standard numerical techniques to solve this type of model is often complicated. In the recent decades, many numerical techniques have been developed and implemented by researchers to solve some aspects of non-classical [7–13]. Specially, the numerical investigation of the two-dimensional diffusion equation with non-classical boundary conditions have been considered by many researchers. Dehghan, has considered and well-used some numerical methods based on the finite difference schemes to numerical investigations of the two-dimensional diffusion equation with non-classical boundary conditions [14–18]. Abbasbandy et al. considered two aspects of the non-classical diffusion equation with Dirichlet and Neumann boundary conditions and have approximated the numerical solution of the problems by using, the local Petrov–Galerkin (MLPG) process based on the Moving

* Corresponding author. Tel.: +98 912 1305326; fax: +98 281 3780040.
*E-mail address:* abbasbandy@yahoo.com (S. Abbasbandy).

0955-7997/$ - see front matter 
http://dx.doi.org/10.1016/j.enganabound.2013.11.005

Least Squares (MLS) approximations [19,20]. Very recently, Kazem and Rad [21] have presented and applied a meshless method based on the radial basis functions to obtain approximate solution of the two-dimensional diffusion equation with Dirichlet and Neumann boundary conditions.

Recently, several meshfree techniques have been attracted great attention and extensively used for numerically solving various types of ordinary and partial differential equations. Meshless techniques are classified into three categories: meshless methods based on strong-forms (collocation methods), meshless methods based on weak-forms and finally meshless methods based on the combination of weak forms and collocation techniques [22]. In the recent decade, several types of meshfree methods based on the radial basis functions (RBFs) such as collocation method based on the radial basis functions (Kansa's method) [23–29] and radial point interpolation method based on the radial basis functions [30–34] have been introduced and developed. The RBF based methods are truly mesh-free methods, which do not require extensive mesh generation and elements, also they are flexible in dealing with problems by irregular domain or multi-dimensional. Recently, a new approach based on the indirect RBF collocation methods has been proposed and improved which use the approximate particular solution of a given differential equation based on the radial basis functions [35–39]. In this work, firstly the method of approximate particular solutions (MAPS), which is based on the strong formulation, is formulated and implemented to numerical investigation of the problem (1.1)–(1.3), then a new aspect of the local radial point interpolation method, which is based on the weak formulation and using particular solutions would be introduced and formulated to deal with the governing problem (1.1)–(1.3).

## 2. Time discretization

For numerical investigation of the governing problem (1.1) with conditions (1.2) and (1.3), firstly a time stepping strategy will be used to discrete the time derivative. Here a general time discretization approach based on the $\theta$-weighted ($0 \le \theta \le 1$) finite difference scheme is applied. By using the $\theta$-weighted scheme, time derivative of the governing problem (1.1) would be discretized at two consecutive time levels $n$ and $n+1$ as follows:

$$\theta\frac{\partial u^{n+1}}{\partial t}+(1-\theta)\frac{\partial u^{n}}{\partial t}=\frac{u^{n+1}-u^{n}}{\delta t}+O(\delta t),$$

where $\delta t = t^{n+1}-t^{n}$ is the time step size and $u^n = u(\mathbf{x}, t^n)$. From the recent relation and Eq. (1.1), the following relation could be concluded:

$$\frac{u^{n+1}-u^{n}}{\delta t}\simeq \theta\alpha\Delta u^{n+1}+(1-\theta)\alpha\Delta u^{n},$$

where $\Delta u = ((\partial^2 u/\partial x^2)+(\partial^2 u/\partial y^2))$ denotes the Laplacian operator. Clearly by rearranging the above relation, one could obtain

$$u^{n+1}-\theta\delta t\alpha\Delta u^{n+1}=u^{n}+\delta t(1-\theta)\alpha\Delta u^{n}. \tag{2.1}$$

In the above implicit time integrated approach, by implementing the initial condition as $u^0 = u(\mathbf{x}, 0)$ the dependent variable at each time level, $\mathbf{u}^n, n \ge 1$, depends on previous time step $\mathbf{u}^{n-1}$. Note that by considering $\theta = 0, 1, 1/2$ in relation (2.1), the resulting equations are equivalent to forward difference, backward difference and Crank–Nicolson methods, respectively.

## 3. Space discretization

In this section two numerical schemes based on the truly meshless methods are proposed to deal with the semi-discretized problem (2.1) and boundary conditions (1.2) and (1.3). Both strong and weak form approaches based on the method of approximate particular solutions (MAPS) and meshless local Petrov–Galerkin (MLPG) method are employed and performed to the problem. In the methods the radial basis functions are employed as trial functions to interpolate the Laplacian operator of the unknown function in the problem (2.1).

### 3.1. The method of approximate particular solutions (MAPS)

In this section a numerical approach based on the method of approximate particular solutions (MAPS) is employed to solve the problem (1.1)–(1.3). Based on the MAPS the Laplacian of the unknown function in $n+1$-th time level in the given strong form of semi-discretized problem (2.1) can be approximated and interpolated at each node in the region of the problem by a linear combination of the radial basis functions ($\varphi(\mathbf{r_j})$) as

$$\Delta\mathbf{u}^{n+1}=\sum_{j=1}^{N}\lambda_j^{n+1}\varphi(\mathbf{r}_j), \tag{3.1}$$

where $\mathbf{r}_j = \{\|\mathbf{x}-\mathbf{x}_j\|, \mathbf{x}_j \in \overline{\Omega}\}$ denotes the Euclidean distance between $\mathbf{x}$ and center points ($\mathbf{x}_j$) and $\{\lambda_j^{n+1}\}_{j=1}^{N}$ is a set of unknown coefficients at $n+1$-th time level to be computed. Through this work the Multiquadrics (MQ) radial basis function is used to interpolate the Laplacian operator in the relation (3.1). The Multiquadrics (MQ) function, $\varphi(\mathbf{r_j}) = \sqrt{\mathbf{r_j}^2 + c^2}$, is the most popular radial basis functions. The positive parameter $c$ appearing in MQ functions is called the shape parameter which dictates the flatness of the multiquadrics function and also all of the MQ based meshless methods have a key role to achieve an accurate and stable scheme. The MQ function has been introduced and employed by Hardy [40], it has infinite smoothness ($MQ \in C^\infty$). Madych [41] has established and presented an error bound as $O(e^{qc}\lambda^{c/\delta})$ for interpolating a smooth function by using multiquadric function, where $0 < \lambda < 1$, $\delta$ is the maximum mesh size, $c$ is the shape parameter and $q$ is a positive constant.

In the current work, the spatial domain is two-dimensional and Laplacian operator can be given as $\Delta = 1/\mathbf{r}(d/d\mathbf{r}(\mathbf{r}(d/d\mathbf{r})))$. Now an approximate particular solution of unknown function at $n+1$-th time level ($\mathbf{u}^{n+1}$) can be easily obtained from (3.1). Indeed the particular solution can be computed by repeated integration of both sides (3.1) [39,42,43] as follows:

$$\begin{aligned}\mathbf{u}^{n+1} &= \int\frac{1}{\mathbf{r}}\int\mathbf{r}\left(\sum_{j=1}^{N}\lambda_j^{n+1}\varphi(\mathbf{r})\right)d\mathbf{r}\,d\mathbf{r}\\ &= \sum_{j=1}^{N}\lambda_j^{n+1}\left(\int\frac{1}{\mathbf{r}}\int\mathbf{r}\varphi(\mathbf{r})\,d\mathbf{r}\,d\mathbf{r}\right)\\ &= \sum_{j=1}^{N}\lambda_j^{n+1}\psi(\mathbf{r}),\end{aligned} \tag{3.2}$$

where $\psi(\mathbf{r})$ is called particular solution correspond to the given radial basis function $\varphi(\mathbf{r})$. For the multiquadrics functions $\varphi(\mathbf{r}) = \sqrt{\mathbf{r}^2+c^2}$ its corresponding particular solution is obtained as

$$\psi(\mathbf{r})=\frac{1}{9}(4c^2+\mathbf{r}^2)\sqrt{c^2+\mathbf{r}^2}-\frac{c^3}{3}\ln(c+\sqrt{c^2+\mathbf{r}^2}).$$

Now, based on the method of approximate particular solutions by substituting the relations (3.1) and (3.2) in Eq. (2.1) and also classical and non-classical boundary conditions (1.2) and (1.3), and then by choosing $N = N_\Omega + N_{\Gamma_1} + N_{\Gamma_2} + N_{\Gamma_3}$ collocation nodes $\{\mathbf{x}_i\}_{i=1}^{N} \in \overline{\Omega} = (\Omega \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ in such a way that $N_\Omega$ and $N_{\Gamma_i}$ denote the number of interior and boundary points of collocation nodes, respectively, the following linear system of equations will