Engineering Analysis with Boundary Elements 59 (2015) 1–7

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Stress analysis for two-dimensional thin structural problems using the meshless singular boundary method

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A R T I C L E I N F O

Article history:
Received 8 August 2014
Received in revised form
24 February 2015
Accepted 28 March 2015
Available online 15 April 2015

Keywords:
Singular boundary method
Method of fundamental solutions
Meshless boundary collocation method
Thin structural problems
Nearly singular kernels
Elasticity problems

A B S T R A C T

This short communication documents the first attempt to apply the singular boundary method (SBM) for the stress analysis of thin structural elastic problems. The troublesome nearly-singular kernels, which are crucial in the applications of the SBM to thin shapes, are dealt with efficiently by using a non-linear transformation technique. Three benchmark numerical examples, ranging from thin films, thin shell-like structures and multi-layer coating systems, are well studied to demonstrate the effectiveness of the proposed method. The advantages, disadvantages and potential applications of the method to thin structural problems, as compared with the boundary element (BEM) and finite element (FEM) methods, are also discussed.

1. Introduction

Most practical problems can be solved effectively simply by recourse to appropriately numerical methods, such as finite element (FEM), finite differences (FDM) or boundary element (BEM) methods [1–4]. Still, there exist large classes of problems for which those traditional methods are not an optimal or even viable option [5,6]. Among these are thin films and thin-layered coatings subjected to spatially distributed forces whose lateral dimensions are large in comparison with the scales of other material dimensions. In these problems, the size and computational effort of a standard numerical model can be very large or even prohibitive, see, for example, Refs. [7,8].

This difficulty is sidestepped herein by means of a hybrid approach that combines the advantages of the BEM with the power of meshless boundary collocation methods [9–15], a numerical tool that is referred to as the singular boundary method, or SBM for short [16]. The main idea is to fully inherit the dimensionality and stability advantages of the former and the meshless and integration-free attributes of the later. This method can be also viewed as one kind of modified method of fundamental solutions (MFS) [17–23], which differs from the traditional MFS in that the source points and collocation points coincide and both are placed on the real boundary directly. Unlike domain discretization methods such as the FEM or FDM, the SBM is a boundary method which means that only the boundary of the solution domain needs to be considered. This makes it particularly attractive for the solution of boundary value problems in which the boundary is of prime interest, such as inverse problems and free boundary problems. Moreover, unlike the BEM, only a collection of boundary nodes is required for the discretization of the problem under investigation in the SBM. These features make the method very easy to implement, in particular for problems in complex geometries and three dimensions. Prior to this study, this method has been successfully tried for 2D problems in potential theory [24] and linear elasticity [16]. Very recently, the method has also been extended to solve 3D problems in potential theory [25,26]. In recent years, a few different numerical methods have been proposed and developed which are different but also related to the method proposed in this paper. The methods developed include, but are not limited to, the isogeometric boundary element method [27–29], scaled boundary method [30], partition of unity method [31], hybrid Trefftz elements method [32,33] and hybrid crack elements method [34]. Furthermore, some interesting remarks of the meshless methods and their engineering applications may be found in the survey paper [35].

This short communication documents the first attempt to apply the SBM for the stress analysis of thin structural elastic problems. The treatment of the nearly singular kernels, which is a crucial step in the application of SBM to thin shapes, is discussed in details. For the text problem studied, promising SBM results with

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http://dx.doi.org/10.1016/j.enganabound.2015.03.017
0955-7997/

only a small number of nodes are obtained with the thickness-to-length ratio of the structure is as small as 10^{-9}, which is sufficient for modeling most thin elastic materials as used in smart materials and micro-electro-mechanical systems (MEMS). A brief outline of the rest of this paper is as follows. The SBM formulation and its implementation are briefly introduced in Section 2. Section 3 introduces a non-linear transformation used in the SBM for solving thin structural problems. Followed in Section 4, the accuracy and efficiency of the proposed method are tested on three benchmark 2D thin structural problems, in which the proposed SBM is compared with the FEM and BEM. Finally, some conclusions and remarks are provided in Section 5.

2. The SBM formulation for 2D elastic problems

The equilibrium equations for 2D problems in linear elasticity, also known as the Navier equations, with respect to the displacement tensor $u_i(\boldsymbol{x})$, $i=1,2$, can be stated as [16]

$$\left\{2\frac{1-\mu}{1-2\mu}\right\}\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_1^2}+\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_2^2}+\left\{\frac{1}{1-2\mu}\right\}\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_1 \partial x_2}=0, \qquad \boldsymbol{x}\in\Omega, \tag{1}$$

$$\left\{\frac{1}{1-2\mu}\right\}\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_1 \partial x_2}+\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_1^2}+\left\{2\frac{1-\mu}{1-2\mu}\right\}\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_2^2}=0, \qquad \boldsymbol{x}\in\Omega, \tag{2}$$

subject to the boundary conditions

$$u_i(\boldsymbol{x})=\overline{u}_i, \qquad \boldsymbol{x}\in\Gamma_u(\text{Dirichlet boundary conditions}), \tag{3}$$

$$t_i(\boldsymbol{x})=\overline{t}_i, \qquad \boldsymbol{x}\in\Gamma_t(\text{Neumann boundary conditions}), \tag{4}$$

where μ is Poisson's ratio, $t_i(\boldsymbol{x})$ denotes the component of the boundary traction in the ith coordinate direction, $\partial\Omega=\Gamma_u+\Gamma_t$ comprises the whole boundary of the domain Ω which we shall assume to be piecewise smooth, $\overline{u}_i$ and $\overline{t}_i$ represent the prescribed displacements and tractions, respectively.

The strains $\varepsilon_{ij}(\boldsymbol{x})$, $i,j=1,2$, are related to the displacement gradients by the kinematic relations

$$\varepsilon_{ij}(\boldsymbol{x})=\frac{1}{2}\left\{\frac{\partial u_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right\}, \tag{5}$$

and the stresses $\sigma_{ij}(\boldsymbol{x})$, $i,j,=1,2$, are related to the strains through Hooke's law by

$$\sigma_{ij}(\boldsymbol{x})=2G\left(\varepsilon_{ij}(\boldsymbol{x})+\frac{\mu}{1-2\mu}\varepsilon_{kk}(\boldsymbol{x})\delta_{ij}\right), \tag{6}$$

where δ_{ij} is the Kronecker delta and G is the shear modulus. The customary standard Cartesian notation for summation over repeated subscripts is used.

The boundary tractions $t_i(\boldsymbol{x})$, $i=1,2$, are defined in terms of the stresses as

$$t_i(\boldsymbol{x})=\sigma_{ij}(\boldsymbol{x})n_j(\boldsymbol{x}), \qquad \boldsymbol{x}\in\partial\Omega, \tag{7}$$

where $n_j(\boldsymbol{x})$ is the direction cosine of the unit outward normal vector at the boundary point $\boldsymbol{x}$.

In the SBM, the displacements and tractions can be approximated by a linear combination of fundamental solutions with respect to different source points $\boldsymbol{x}$ as follows

$$u_i(\boldsymbol{y}^m)=\sum_{n=1} m\neq n^N \alpha_j^n U_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n)+\alpha_j^m A_{ij}^m, \qquad \boldsymbol{y}^m\in\Gamma_u, \tag{8}$$

$$t_i(\boldsymbol{y}^m)=\sum_{n=1} m\neq n^N \alpha_j^n T_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n)+\alpha_j^m B_{ij}^m, \qquad \boldsymbol{y}^m\in\Gamma_t, \tag{9}$$

where $i,j=1,2$, $\{\alpha_j^n\}_{n=1}^N$ represent the unknown coefficients, $\boldsymbol{y}^m\in\overline{\Omega}=\Omega\cup\partial\Omega$ is the mth collocation point, $\boldsymbol{x}^n$ stands for the nth source point, the second order matrixes A_{ij}^m and B_{ij}^m are defined as the *origin intensity factors*, i.e., the diagonal and sub-diagonal elements of the SBM interpolation matrix. It is interesting to note that the proposed SBM sidesteps the troublesome *fictitious boundary issue* associated with the traditional MFS by means of the introduction of the aforementioned origin intensity factors, a numerical strategy that isolate the singularities of the fundamental solutions. In Eqs. (8) and (9), U_{ij} and T_{ij} represent the displacement and traction fundamental solutions for 2D elastic problems. Their explicit expressions can be found in Ref. [36].

The accurate evaluation of origin intensity factors plays a key role in the present method. Quite recently, Gu and Chen [37] have provided an efficient algorithm for the direct calculation of the above-mentioned origin intensity factors in the SBM formulation. The proposed method can be successfully applied to 2D and 3D SBM formulations, regardless of the specific class of problems being considered and whatever the type and order of the singular terms encountered. For completeness, the main results for 2D elastic problems are summarized hereafter. Details on the derivations of some of these formulas can be found in Refs. [36,37]. The origin intensity factors B_{ij}^m for Neumann boundary conditions (9) can be expressed as

$$B_{ij}^m=\frac{1}{L_m}\left[\tau\delta_{ij}+I_m+\sum_{n=1} n\neq m^N T_{ij}(\boldsymbol{x^n},\boldsymbol{y^m})L_n\right], \tag{10}$$

where L_m is the half distance between the $(m-1)$th and $(m+1)$th nodes, I_m is a regular function [37], and the symbol τ is defined as

$$\tau=\begin{cases}1, & \text{for interior problems,}\\ 0, & \text{for exterior problems.}\end{cases} \tag{11}$$

The origin intensity factors A_{ij}^m for Dirichlet boundary conditions (8) present a weak singularity of order ($\ln r$), which can be directly set as an average value of the fundamental solution over a portion of the boundary. Algorithms relating to the direct evaluation of weakly singular integrals are widely available and more details can be found, e.g., in Refs. [18,38]. Alternatively, the origin intensity factors A_{ij} can be calculated indirectly using the so-called "inverse interpolation method" proposed in Ref. [16].

For a well-posed boundary value problem, the unknown coefficients $\{\alpha_j^n\}_{n=1}^N$ can be determined by collocating N observation points on the boundary conditions from Eq. (8) for Dirichlet problems and Eq. (9) for Neumann problems. Once all coefficients are computed, the displacements and stresses at any point inside the domain can be obtained directly using the following strong-form formula

$$u_i(\boldsymbol{y})=\sum_{n=1}^N \alpha_j^n U_{ij}(\boldsymbol{y},\boldsymbol{x}^n), \tag{12}$$

$$\sigma_{ij}(\boldsymbol{y})=\sum_{n=1}^N \alpha_k^n D_{ijk}(\boldsymbol{y},\boldsymbol{x}^n), \tag{13}$$

where $\boldsymbol{y}\in\Omega$, D_{ijk} are fundamental solutions for stresses [36].

3. The transformed SBM for thin structural problems

The numerical difficulty in the SBM is the nearly-singular kernels which arise in both crack-like and thin structural problems. In such cases, the nodes on one side of the boundary usually being too close to the nodes on the opposite side, leading to the kernels present various orders of near-singularities. The nearly