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Effectiveness of nonsingular solutions of the boundary problems based on Trefftz methods

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ABSTRACT

The paper describes the application of the Trefftz complete and Kupradze functions in two variational formulations, i.e. the original formulation and inverse one, to the solution of the boundary value problems of the two-dimensional Laplace's equation. In both formulations the solutions and weighting functions are assumed as the series or the separate function of Trefftz complete functions or Kupradze ones. One way or another all methods are named Trefftz methods. They all are nonsingular and, at the same time, they lead to the BEM. The relationship between the groups of Trefftz methods of the original and inverse formulations is perceived.

Numerical experiments are conducted for several Laplace problems. The accuracy and simplicity of the methods are discussed. All methods gave comparable results, therefore they may be interchangeably applied to the solution of boundary problems. However the best method group is pointed out.

1. Introduction

Boundary element method (BEM) has certain computational advantages in a class of linear problems. In particular, it permits a simple treatment of infinite domains where full field methods such as finite element method or finite difference method are difficult to operate [1,2]. Additionally, an economy is that of data preparation, i.e. it requires a geometrical definition of the boundaries only and not of the mesh within.

One considerable disadvantage of the BEM, the conventional one, in which the singular fundamental solution is used, is the difficulty encountered in the use of numerical integration over singularities. This disadvantage can be overcome in many ways; an original one is proposed in [3]. This paper presents an improvement of the singular boundary method of the Helmholtz equation boundary problem. In order to eliminate the singularity, it is introduced the concept of source intensity factors. The concept proposed to the same order of the singularities of fundamental solutions of the Laplace equation [4,5] is expanded.

If the solution of the boundary problem is assumed as the series, the next way is to choose the base functions (bases) so as the solution satisfies the (homogeneous) governing equations; it leads directly to BEM. This requirement meets broadly comprehended Trefftz functions and it immediately generates the Trefftz methods (TMs), originally formulated in [6]. The unknown coefficients are determined by matching the boundary conditions. Hereafter, as Trefftz functions (*T*-functions) are defined the *T*-complete functions described in [7,8]. The mathematical theory of TMs is developed in [7,9–11].

Conventionally, TMs are divided into two main groups, i.e. the direct formulation and indirect one; see references given in [12]. The former, [12–16], is derived from inverse variational formulation. So, the *T*-functions are taken as the weighting functions (weights). Since the weights satisfy the differential equation, one obtains the boundary integral equation (BIE), from which the suitable unknown coefficients are obtained.

The latter group, [17–20], is derived from either classical formulation or original variational one. In this case, the solution is approximated by the series of the *T*-functions with unknown coefficients, which are determined so that the approximate solution satisfies boundary conditions and it also leads to BIE. Since both direct and indirect formulations lead to the BIE so TMs are BEM. As mentioned above, an excellent survey of the BEM is given in [1,2]. But the survey does not include details and TMs are not perceived as separate method. The gap is filled by [21,22] where the survey of TMs is given. Furthermore the comparison among TMs and other numerical methods is made and the review of coupling techniques which includes TM and its variants and versions are run.

The second such method, which leads to BEM, is a method of fundamental solutions (MFS). The MFS can be reviewed as an

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indirect BEM with concentrated sources instead of distribution. The initial idea is to approximate the solution with a linear combination of fundamental solutions with sources located outside the domain; such solutions are proposed in [11] and references cited therein, so hereafter they are called as *K*-functions.

The link between TMs and MFS is detailed discussed in [23,24]. They are compared in classical formulation. In [24] one focuses on providing the equivalence on the numerical Green's functions for annular Laplace problem derived by using the TM and the image method (image method is a special case of MFS). The two solutions of TMs and MFS were proved to be mathematically equivalent by using additional theorem or so-called degenerate kernel.

The least square method (LSM) and ultraweak variational formulation are compared in the TMs in [25], in which the solution is assumed as the series of the *T*-functions. In that paper, in the framework of LSM, the quadratic functional (an energy) is formulated. Minimizing the functional, the constant coefficients are calculated. This method ought to be included into the energetic methods rather (variational ones) or into the weak variational formulation and Galerkin version. Using firstly weighted residual method, quite the same attitude is applied in the ultraweak variational formulation.

An expanded survey of some TMs is given in [21,22]. In those works, the TM, the collocation method, and the collocation TM are compared. Furthermore, the hybrid methods, which include the indirect TM, the original one, the penalty plus hybrid TM, and the direct TM are reviewed for the inter-zonal conditions. Other boundary methods are also briefly described. Comparisons among TMs and other numerical methods are made.

None of the above papers contains the cross comparison of the nonsingular boundary methods basing on the assumed a priori nonsingular bases; such bases constitute *T*-functions and *K*-functions. Sometimes all these methods are named commonly as TMs.

In this work this gap is filled. The nonsingular methods are derived from the original and the inverse variational formulations; some results are published in [26,27]. Furthermore, they are in the form of the series in which the bases are either *K*-functions or *T*-functions. Furthermore the weights take the same forms as the bases. To sum up, each separate method contains *T*-functions and *K*-functions as the bases or as the weights. This way eight separate methods are formulated and all lead to the first kind of Fredholm BIE. It seems that all nonsingular solutions of Laplace's boundary problems based on *T*-functions and *K*-functions are formulated. In order to distinguish the detailed differences among them, the appropriate symbols are introduced.

The methods are first formulated. Since the Laplace's equation is the simplest one, thus it is utilized for studying robustness and efficiency of the derived TMs. Numerical experiments are provided for three boundary problems with analytical solutions. The results are depicted in figures and furthermore the method errors, via Euclidean norm, are quantitatively given. In the end some conclusions are derived.

2. Kinds of boundary problem formulations

A general formulation of boundary problems is the classical formulation and the variational one.

2.1. Classical formulation of boundary problems

Let in the domain Ω surrounded by the boundary Γ be given the boundary problem described by a differential equation and the boundary condition, Fig. 1. The Γ is divided into two main elements, i.e. $\Gamma = \cup_j \Gamma_j$, $\Gamma_j = \Gamma_u$ or Γ_v. Let $\Delta = L$ be Laplace

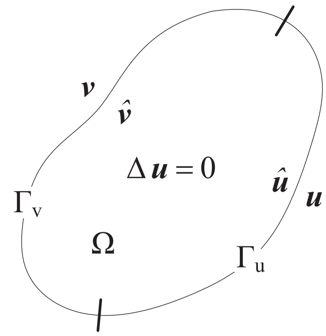

Fig. 1. Geometry of the general boundary problem.

operator, hence the boundary problem is as follows

$$\begin{aligned} &\Delta u(\mathbf{x}) = Lu(\mathbf{x}) = 0, && \mathbf{x} = \dot{\mathbf{x}} \in \Omega, \\ &u(\mathbf{x}) = \hat{u}(\mathbf{x}), && \mathbf{x} \in \Gamma_u \\ &D_{\mathbf{n}} u(\mathbf{x}) = v(\mathbf{x}) = \hat{v}(\mathbf{x}), && \mathbf{x} \in \Gamma_v \end{aligned} \tag{1}$$

where $\hat{u}(\mathbf{x})$, $\hat{v}(\mathbf{x})$ are given functions, the rest of symbols is depicted in Fig. 1.

The problem can be rewritten in a compact form

$$Du(\mathbf{x}) = h(\mathbf{x}), \qquad \mathbf{x} \in \overline{\Omega} \tag{2}$$

where $D = L/B$ and B is the boundary operator; it represents Dirichlet or Neumann boundary conditions, $h(\mathbf{x}) = 0 \,/\, g(\mathbf{x})$ and $\overline{\Omega} = \Omega \cup \Gamma$.

The boundary problem, described by Eq. (2), is classically formulated. Substituting the approximate solution $\tilde{u}(\mathbf{x})$, instead exact one $u\,(\mathbf{x})$ into Eq. (2), this equation is not satisfied and some residuum is generated, hence

$$E_{\overline{\Omega}}(\mathbf{x}) = D\tilde{u}(\mathbf{x}) - h(\mathbf{x}) \neq 0, \qquad \mathbf{x} \in \overline{\Omega} \tag{3}$$

where $E_{\overline{\Omega}}(\mathbf{x}) = E_\Omega(\mathbf{x}) \,/\, E_\Gamma(\mathbf{x})$.

This form of the boundary problem is the starting point for the variational formulations.

2.2. Variational formulations of boundary problems

The transformation of the classical formulation into variational ones is done via the weighted residual method (WRM) [28–30]. The first step of WRM is multiplying Eq. (3) by any weight $W_{\overline{\Omega}}(\mathbf{x}) = W_\Omega(\mathbf{x})/W_\Gamma(\mathbf{x})$. Hence, the weighted equation is obtained

$$E_{\overline{\Omega}}(\mathbf{x}) W_{\overline{\Omega}}(\mathbf{x}) \neq 0, \qquad \mathbf{x} \in \overline{\Omega} \tag{4}$$

Integrating Eq. (4) over domain $\overline{\Omega}$ gives the weighted integral equation

$$\int_{\overline{\Omega}} E_{\overline{\Omega}}(\mathbf{x}) W_{\overline{\Omega}}(\mathbf{x}) d\mathbf{x} \neq 0, \qquad \mathbf{x} \in \overline{\Omega} \tag{5}$$

An idea of WRM lies in such distribution of the residuum $E_{\overline{\Omega}}(\mathbf{x})$ in domain $\overline{\Omega}$, via selection of the weight $W_{\overline{\Omega}}(\mathbf{x})$, so that the weighted integral equation should be equal to zero, namely

$$\int_{\overline{\Omega}} E_{\overline{\Omega}}(\mathbf{x}) W_{\overline{\Omega}}(\mathbf{x}) d\mathbf{x} = 0, \qquad \mathbf{x} \in \overline{\Omega} \tag{6}$$

Eq. (6) is the basic one of WRM; sometimes it is called weighted residual statement. At the same time, it is an original variational formulation of the boundary problem Eq. (2).

In practice, it is convenient to instill boundary conditions into variational formulations. To this purpose, one can write Eq. (6) in explicit form as

$$\int_\Omega L\tilde{u}\; W\, d\mathbf{x} = \int_\Omega \Delta\tilde{u}\; W\, d\mathbf{x} = 0 \tag{7}$$