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Recovery of the temperature and the heat flux by a novel meshless method from the measured noisy data

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ABSTRACT

In this paper, we give an invariant method of fundamental solutions (MFS) for recovering the temperature and the heat flux. The invariant MFS is to keep a very basic natural property, which is called the invariance property under trivial coordinate changes in the problem description. The optimal regularization parameter is chosen by Morozov discrepancy principle. Then the reason for introducing the regularization is explained clearly by using the potential function. Three kinds of boundary value problems are investigated to show the effectiveness of this method with some examples. In especial, when the classical MFS does not give accurate results for some problems, it is shown that the proposed method is effective and stable. For each example, the numerical convergence, accuracy, and stability with respect to the number of source points, the distance between the pseudo and real boundary, and decreasing the amount of noise added into the input data, respectively, are also analyzed.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with piecewise smooth boundary $\partial\Omega$, Γ_1 be a portion of $\partial\Omega$, and set $\Gamma_2 = \partial\Omega \backslash \overline{\Gamma}_1$. In this paper, we refer to steady-state heat conduction applications in an isotropic homogeneous media, and consider the interior problem for the Laplace equation. Consequently, in the absence of heat sources, the function $u(\boldsymbol{x})$ denotes the temperature at a point $\boldsymbol{x} \in \Omega$, satisfies the steady-state heat equation

$$\Delta u = 0 \quad \text{in } \Omega. \tag{1}$$

Now let $n(\boldsymbol{x})$ be the unit outward normal vector on $\partial\Omega$, and $q(\boldsymbol{x})$ be the normal heat flux at a point $\boldsymbol{x} \in \partial\Omega$ defined by

$$q(\boldsymbol{x}) = \nabla u \cdot n(\boldsymbol{x}). \tag{2}$$

In this paper, we consider (1) combining the following boundary conditions:

- Dirichlet boundary condition (given the temperature)

$$u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{on } \partial\Omega. \tag{3}$$

- Neumann boundary condition (given the normal heat flux)

$$q(\boldsymbol{x}) = \tilde{q}(\boldsymbol{x}) \quad \text{on } \partial\Omega. \tag{4}$$

- Mixed boundary condition (given the partial temperature and the partial normal heat flux)

$$\begin{cases} u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{on } \Gamma_1, \\ q(\boldsymbol{x}) = \tilde{q}(\boldsymbol{x}) & \text{on } \Gamma_2. \end{cases} \tag{5}$$

One popular method for solving these problems is the MFS. The MFS, first introduced by Kupradze and Aleksidze [1], is a well-known meshless method. It is an effective method to deal with the direct and inverse problems governed by the partial differential equations. Its numerical formulation was first given by Mathon and Johnston [2]. The main idea of the MFS is to approximate the solution by a linear combination of fundamental solutions with respect to some source points located outside the solution domain.

This paper is to recover the temperature and the heat flux by an invariant method of fundamental solutions (IMFS). The main idea is to approximate the solution by the MFS with invariant condition, which is a solution of the Laplace equation (1) of the form

$$u(\boldsymbol{x}) = c + \sum_{j=1}^{N} a_j \Phi(\boldsymbol{x}, \boldsymbol{y}_j), \qquad \sum_{j=1}^{N} a_j = 0, \tag{6}$$

where $\boldsymbol{y}_j \in \mathbb{R}^2 \backslash \overline{\Omega}$, c is a constant, and a_j are the coefficients. Here, $\Phi(\boldsymbol{x}, \boldsymbol{y}) = (1/2\pi)\ln|\boldsymbol{x} - \boldsymbol{y}|$ is the fundamental solution.

The MFS, also named the charge simulation method, is a meshless method. It is extremely attractive to solve the problems with complicated boundary. The method has become increasingly

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popular because of the simple implementation, especially for the problems in complex geometries. The simple implementation of the MFS for the problems with complex boundaries makes it an ideal candidate for the problems in which the boundary is of major importance or requires special attention. For these reasons, the MFS has been used increasingly over the last decade for the numerical solution of various problems, including the inverse problems [3] and the direct problems [4,5]. Fairweather and Karageorghis [3] and Karageorghis et al. [6] presented the excellent surveys of the MFS and related methods over the past three decades. As marked in [3], the MFS was used to solve a variety of more complicated problems such as plane potential problems involving nonlinear radiation-type boundary conditions, free boundary problems, biharmonic problems, elastostatics problems and wave scattering problems. The MFS is also used to solve the unsteady state heat equation, see [7] for details.

Although the MFS has been used to deal with the numerical solutions of various problems, it should be noted that it is possible to obtain an inaccurate solution for using the classical MFS. As marked by Chen et al. [8], many researchers proposed to enrich their formulations. Alves and Leitao [9] introduced an enriched MFS to simulate a crack singularity. Saavedra and Power [10,11] added a constant term in their formula, and marked that this is for the completeness purposes in two-dimensions. Although the constant in the MFS is sometimes recommended, it is rarely used in practice as the degenerate cases occurring rather rarely. In the literatures, the inclusion of the constant has been discussed by several researchers [12–15]. In order to avoid the trouble of choosing a fictitious boundary, Chen et al. [16] developed the singular boundary method (SBM). They assumed a test example to calculate the source weighting, and then used this source weighting to determine the value of diagonal term where the source and field points can coincide. However, for certain case, the SBM yields an inaccurate approximation. As a result, they provided an improved formulation of adding a constant term and a constraint [17,18]. They called this constraint “moment condition”. However, the role of the constant and the constraint were not discussed in detail. Although they [16–20] have successfully solved the problems, they did not deeply examine the role of the constant term and the constraint. Following Fichera's idea, Chen et al. [8] enriched the MFS, formulation the MFS by an added constant and a constraint, which can be used to solve not only the interior problem, but also the exterior case. They also explained that the constant term and the constraint were required to ensure a unique solution for the degenerate scale. Besides, the role in exterior problems is also examined. But they do not investigated the corresponding enriched MFS for the Neumann boundary condition and the mixed boundary condition. The numerical study of this method to solve these three problems is our main goal. In a recent paper by Chen et al. [21], they added a free constant and an extra constraint to enrich the indirect boundary element method formulation to overcome the incompleteness of the indirect boundary element method for the interior problem containing a degenerate scale and for the exterior problem with a bounded potential at infinity in their paper. The enriched indirect boundary element method can be used not only for the interior 2D problem in the case of a degenerate scale but also the exterior problem with a bounded potential at infinity. Furthermore, the constant term is added to compensate for the range deficiency by a constant field in case of a degenerate scale. For a degenerate scale, we can refer to [22–26]. In [27], Sun and Ma have given a new view of this equation and derived it by using the invariance argument. The authors prove that the invariance property is the essence of the analytical solution. It should be noted that the degenerate scale and the invariance property both will give the free constant and an extra constraint for the MFS. A clear linking on connections of the MFS, Trefftz method, indirect boundary integral equation method and invariant MFS was established by Chen et. al [28].

The outline of this paper is as follows. In Section 2, we firstly formulate the IMFS, then give an invariant MFS to solve the heat conduction problem. In Section 3, we will explain the necessary of the regularization for solving this boundary value problem. Finally, five numerical examples are included to show the effectiveness of the method for solving three kinds of boundary value problems. The first one is a comparative study between the classical MFS and the IMFS. The second one is a problem with Dirichlet boundary condition on a simply connected domain. The third one is a problem with mixed boundary condition on a doubly connect domain. The forth one is a problem with Neumann boundary condition on a non-convex peanut domain. The last one is a problem with mixed boundary condition on a square domain.

2. Formulation of the IMFS

Let us consider the following two simple problems: let D_1 and D_2 be two simply connected domains with smooth boundaries. For every $\boldsymbol{x} \in \partial D_1$, there is $\boldsymbol{x}' = \lambda \boldsymbol{x} \in \partial D_2, \lambda > 0$. Let u and u' satisfy

- (P.1)

$$\begin{cases} \Delta u = 0 & \text{in } D_1, \\ u = f & \text{on } \partial D_1. \end{cases}$$

- (P.2)

$$\begin{cases} \Delta u' = 0 & \text{in } D_2, \\ u' = f' & \text{on } \partial D_2. \end{cases}$$

If we give the boundary conditions $f(\boldsymbol{x}) = f'(\lambda \boldsymbol{x})$ for $\boldsymbol{x} \in \partial D_1$, we know that the two solutions have some relevance, i.e., $u(\boldsymbol{x}) = u'(\lambda \boldsymbol{x})$ for $\boldsymbol{x} \in D_1$. We call that the *invariance property* under trivial coordinate changes in the problem description as scaling of coordinates.

The classical MFS assumes an approximation of u by the following form

$$u_C(\boldsymbol{x}) = \sum_{j=1}^{N} a_j \Phi(\boldsymbol{x}, \boldsymbol{y}_j), \tag{7}$$

where $\boldsymbol{y}_j \in \mathbb{R}^2 \setminus \overline{\Omega}$ are the source points, a_j are the coefficients which will be determined from the boundary condition.

However, the approximation $u_C(\boldsymbol{x})$ constructed in this way lacks an essential property [27], i.e. the invariance under trivial coordinate changes in the problem description such as scaling of coordinates:

$$\boldsymbol{x} \to \lambda \boldsymbol{x}, \quad \boldsymbol{y}_j \to \lambda \boldsymbol{y}_j. \tag{8}$$

To be more specific, we expect that the approximation $u_C(\boldsymbol{x})$ of $u(\boldsymbol{x})$ should transform as

$$u_C(\boldsymbol{x}) \to u'_C(\lambda \boldsymbol{x}) \tag{9}$$

under the transformations (8). Thus we can simultaneously get the solutions of (**P.1**) and (**P.2**). However, $u_C(\boldsymbol{x})$ is not the case. Since

$$\begin{aligned} u'_C(\lambda \boldsymbol{x}) &= \sum_{j=1}^{M} a_j \Phi(\lambda \boldsymbol{x}, \lambda \boldsymbol{y}_j) = \sum_{j=1}^{M} a_j \ln |\lambda \boldsymbol{x} - \lambda \boldsymbol{y}_j| \\ &= \sum_{j=1}^{M} a_j \ln |\lambda \boldsymbol{x} - \lambda \boldsymbol{y}_j| = u_C(\boldsymbol{x}) + \sum_{j=1}^{M} a_j \ln \lambda, \end{aligned} \tag{10}$$

and in general, $\sum_{j=1}^{M} a_j \neq 0$. Thus, $u'_C(\lambda \boldsymbol{x}) \neq u_C(\boldsymbol{x})$.