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Non-singular Method of Fundamental Solutions for anisotropic elasticity

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ABSTRACT

The purpose of the present paper is to develop a Non-singular Method of Fundamental Solutions (NMFS) for two-dimensional anisotropic linear elasticity problems. The NMFS is based on the classical Method of Fundamental Solutions (MFS) with regularization of the singularities. This is achieved by replacing the concentrated point sources with distributed sources over disks around the singularity, as recently developed for isotropic elasticity problem. In case of the displacement boundary conditions, the values of distributed sources are calculated by a simple numerical procedure, since the closed form solution is not available. In case of traction boundary conditions, the respective desingularized values of the derivatives of the fundamental solution in the coordinate directions, as required in the calculations, are calculated indirectly by considering two reference solutions of the linearly varying simple displacement fields. The feasibility and accuracy of the newly developed method are demonstrated through comparison with MFS solutions and analytical solutions for a spectra of anisotropic plane strain elasticity problems, including bi-material arrangements. NMFS turns out to give similar results as the MFS in all spectra of performed tests. The lack of artificial boundary is particularly advantageous for using NMFS in multi-body problems.

1. Introduction

In recent years, there has been a strong development of mesh reduction methods in which polygon-like meshes are reduced or avoided. The Method of Fundamental Solutions (MFS) (sometimes also called the F-Trefftz method, the charge simulation method, or the singularity method [1–3]) is a numerical technique that belongs to the class of methods generally called boundary methods. The other well-known representative of these methods is the boundary element method (BEM) [4,5]. Both methods are best applicable in situations where a fundamental solution of the partial differential equation in question is known. In such cases, the dimensionality of the discretization is reduced. The BEM, for example, requires polygonization of the boundary surfaces in general three-dimensional (3D) cases, and boundary curves in general 2D cases. This BEM approach requires the solution of complicated regular, weakly singular, strongly singular, and hyper-singular integrals over boundary segments which is usually quite a cumbersome and a non-trivial task. The MFS [6] has certain advantages over BEM that stem mostly from the fact that only the pointization of the boundary is needed, which completely avoids any integral evaluations, and makes no principal difference in coding between the 2D and the 3D cases. On the other hand, when a singular fundamental solution is involved, the MFS requires nodes that are positioned on an artificial boundary located outside the computational domain. The location of the artificial boundary represents the most challenging problem of the MFS and has to be dealed with heuristically [7]. The expansion coefficients of the solution in MFS are determined so that the solution satisfies the boundary condition with the help of direct collocation [6,7], least squares approximation [8], or by an integral fit of the boundary data [9,10]. Moreover, it has certain advantages over BEM, e.g. no singularity and no boundary integrals are required. Both BEM and MFS are ideal candidates for solving anisotropic linear elasticity problems [11], since the fundamental solution for this type of problems is known [12,13].

In recent years, various efforts have been made, aiming to remove this drawback of the MFS, so that the source points can be placed on the real boundary directly [14–16]. They introduce novel ideas to determine the diagonal collocation matrix coefficients. In [14–16], the diagonal coefficients were determined directly for simple geometries or by using the results from the BEM, based on

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the fact that the MFS and the indirect boundary integral formulation are similar in nature. In these approaches, information of the neighboring points before and after each source point is needed, in order to form line segments for integrating the kernels to obtain the diagonal coefficients. This is essentially the same information of the element connectivity as in BEM. In [17], a similarly modified MFS is proposed, where the diagonal terms are determined by the integration of the fundamental solution on the line segments formed by using neighboring points, and the use of a constant solution to determine the diagonal coefficients of the derivatives of the fundamental solution in different coordinate directions. This approach is very stable, but it amounts to solve the problem twice. In [18], a similar method was developed for determining the diagonal coefficients in the modified MFS by applying a known solution inside the domain, so that the diagonal coefficients from both the fundamental solution and its derivative can be determined indirectly, without using any element or integration concept. Again, this approach is appealing, stable, and accurate but it is costly for solving large-scale problems due to the need to solve the problem twice. The solution also depends on the choice of the reference points. In [19], a singular boundary method was applied to two-dimensional (2D) elasticity problems, in which the authors use an inverse interpolation technique to regularize the singularity of the fundamental solution of the equation governing the problem of interest. A regularized meshless method was also developed for the non-homogeneous problems [20] in connection with the dual reciprocity technique in the evaluation of the particular solution. Liu [21] recently presented a new boundary meshfree approach based on the modified MFS that has no fictitious boundaries and singularities. In the new approach, the concentrated point sources are replaced by area-distributed sources covering the source points for 2D problems. These area-distributed sources represent analytical integration of the original singular fundamental solution, so that they preserve the advantage of diagonal dominance for the system of equations, while they have no troublesome singularity issues. Liu [21] gave the method the name Boundary Distributed Source (BDS) method. In [21], the author used the approach shown in [17] to determine the diagonal coefficients of the derivatives of the fundamental solution. Liu's approach [21] has been recently used to solve porous media problems with moving boundaries in [22]. A review paper on non-singular MFS has been recently published by Gaspar [23].

The plane elasticity theory of isotropic materials has been well established [24]. Both the stress and displacement formulations have been successfully applied to solve various problems [25,26]. On the other hand, the theory of planar anisotropic elasticity, which deals with the classical plan stress and plane strain problems of anisotropic elastic materials possessing one plane of symmetry at $x_3=0$ (i.e., monoclinic elastic materials), is still an active research topic [27]. There are six independent elastic constants in a problem of planar anisotropic elasticity, as opposed to two in a problem of planar isotropic elasticity. Detailed discussions and general formulas are provided in [28] for problems of planar anisotropic elasticity. The MFS for anisotropic elasticity problems was considered in [29,30]. Liu's approach [21] for Non-singular MFS of isotropic elasticity problems, was proposed by Liu and Šarler [31] recently. Its application to general anisotropic and non-linear problems might not be as effective as to linear isotropic problems. The main reason is that the anisotropy of a material increases the number of material properties, and hence makes the fundamental solutions either too complex or unavailable in a closed form [32–34].

In the present paper, we use a Non-singular MFS (NMFS), based on [19,29], to deal with the two dimensional (2D) anisotropic elasticity problems. We respectively use the area-distributed sources covering the source points to replace the concentrated point sources. This NMFS approach also does not require a detailed information about the neighboring points for each source point, thus it is a truly meshfree boundary method. The derivatives of the fundamental solution in the distributed source points are calculated by adopting the methodology in [17] from the Laplace to anisotropic elasticity solution. The rest of the paper is structured as follows. Solution procedure is elaborated for MFS and NMFS in a uniform way. Numerical examples of different type of deformations with analytical solutions are presented to demonstrate the feasibility and accuracy of the NMFS, followed by bi-material examples. At the end, the conclusions and further research directions are given.

2. Governing equation

We begin our discussion with three dimensional (3D) problems and then reduce it to 2D problems of interest. Consider a 3D domain Ω with the boundary Γ filled with anisotropic elasticity materials. Let us introduce a 3D Cartesian coordinate system with orthonormal base vectors $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ and coordinates p_x, p_y and p_z of position vector $\mathbf{p}$, i.e. $\mathbf{p}=p_x\mathbf{i}_x+p_y\mathbf{i}_y+p_z\mathbf{i}_z$. In order to simplify the discussion, we shall assume (i) the solid is free of body forces, and (ii) the thermal strains can be neglected. Under these conditions the general equation of elasticity [35] is

$$C_{\varsigma\xi\upsilon\tau}\frac{\partial^2 u_\upsilon(\mathbf{p})}{\partial p_\varsigma\,\partial p_\tau}=0,\quad \varsigma,\xi,\upsilon,\tau=x,y,z, \tag{1}$$

where u_υ are the displacements, and $C_{\varsigma\xi\upsilon\tau}$ are the components of a fourth rank stiffness tensor [12]

$$\mathbf{C}\equiv C_{\varsigma\xi\upsilon\tau}=\begin{bmatrix} C_{xxxx} & C_{xxyy} & C_{xxzz} & C_{xxyz} & C_{xxxz} & C_{xxxy}\\ C_{xxyy} & C_{yyyy} & C_{yyzz} & C_{yyyz} & C_{yyxz} & C_{yyxy}\\ C_{xxzz} & C_{yyzz} & C_{zzzz} & C_{zzyz} & C_{zzxz} & C_{zzxy}\\ C_{xxyz} & C_{yyyz} & C_{zzyz} & C_{yzyz} & C_{yzxz} & C_{yzxy}\\ C_{xxxz} & C_{yyxz} & C_{zzxz} & C_{yzxz} & C_{xzxz} & C_{xzxy}\\ C_{xxxy} & C_{yyxy} & C_{zzxy} & C_{yzxy} & C_{xzxy} & C_{xyxy} \end{bmatrix}$$
$$=\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix}. \tag{2}$$

The stresses $\sigma_{\varsigma\xi}$ are related to the strains through generalized Hooke's law

$$\boldsymbol{\sigma}=\boldsymbol{C}\boldsymbol{\varepsilon}, \tag{3}$$

where $C_{\varsigma\xi\upsilon\tau}$ satisfy the full symmetry conditions

$$C_{\varsigma\xi\upsilon\tau}=C_{\xi\varsigma\upsilon\tau},\ C_{\varsigma\xi\upsilon\tau}=C_{\varsigma\xi\tau\upsilon},\ C_{\varsigma\xi\upsilon\tau}=C_{\upsilon\tau\varsigma\xi}. \tag{4}$$

$\boldsymbol{\varepsilon}$ is vector of strains

$$\boldsymbol{\varepsilon}\equiv\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 2\varepsilon_{yz}\\ 2\varepsilon_{xz}\\ 2\varepsilon_{xy}\end{bmatrix}=\begin{bmatrix}\partial u_x/\partial p_x\\ \partial u_y/\partial p_y\\ \partial u_z/\partial p_z\\ \partial u_y/\partial p_z+\partial u_z/\partial p_y\\ \partial u_x/\partial p_z+\partial u_z/\partial p_x\\ \partial u_x/\partial p_y+\partial u_y/\partial p_x\end{bmatrix}. \tag{5}$$

For plane strain problems, $\varepsilon_{zz}=\varepsilon_{xz}=\varepsilon_{yz}=0$. Under these conditions, the equilibrium equations reduce to

$$c_{11}\frac{\partial^2 u_x}{\partial {p_x}^2}+c_{66}\frac{\partial^2 u_x}{\partial {p_y}^2}+2c_{16}\frac{\partial^2 u_x}{\partial p_x\,\partial p_y}+c_{16}\frac{\partial^2 u_y}{\partial {p_x}^2}+c_{26}\frac{\partial^2 u_y}{\partial {p_y}^2}$$