Engineering Analysis with Boundary Elements 55 (2015) 52–57

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

An iterative coupling between meshless methods to solve embedded crack problems

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ARTICLE INFO

Article history:
Received 2 December 2013
Accepted 19 February 2014
Available online 15 December 2014

Keywords:
Iterative coupling
Meshless
Green's function
Crack

ABSTRACT

A truly meshless iterative coupling is presented to solve linear elastic fracture mechanic (LEFM) problems. The global domain of the problem is decomposed into sub-domains, where each one is addressed using an appropriate meshless method. The sub-domain which has embedded cracks is modeled by the method of fundamental solutions (MFS) with the help of the numerical Greens function (NGF) approach and the sub-domain without cracks is modeled by the meshless local Petrov–Galerkin (MLPG) procedure. By using the NGF approach the representation of the crack is automatically included. The specific computations of each meshless method are performed independently, coupled with an iterative renewal of variables procedure, restricted to interface unknowns, to achieve the final convergence. The iterative solution procedure presented yields good results as compared with the boundary element method and analytical solutions for stress intensity factor computations.

1. Introduction

Meshless methods have been increasingly applied to obtain solutions of partial differential equations as an alternative to mesh type methods like the boundary element method (BEM) and the finite element method (FEM). Previous references on the development of alternative meshless methods can be found in the works by Belytschko et al. [1], Golberg and Chen [2], Atluri [3], and Nguyen et al. [4]. However like in any numerical approach, meshless methods can present inherent drawbacks depending on the engineering problem to be solved. To circumvent this, coupling procedures using the appropriate meshless methods for each typical problem region can be adopted, improving not only efficiency, but also solution accuracy for different coupled engineering problems as shown in the works [5–7].

The method of fundamental solutions (MFS), developed by Kupradze and Aleksidze [8] is a simple meshless boundary-type method. In order to build the solution, MFS uses a superposition of fundamental solutions associated with the problem. This is done without using any integrals, greatly simplifying its implementation. This advantage actually exposes the drawback of the MFS, which is the necessity of the location of the virtual sources to generate a good solution. For linear elastic fracture mechanics (LEFM) problems a regularized version of the MFS-NGF was developed which greatly facilitated the positioning of the virtual sources and to obtain a stabilized solution for the system of linear equations, improving the accuracy, as shown in the work [9]. Additionally by using the NGF approach, the presence of cracks is already built-in in the formulation. So the NGF procedure can be seen to be of great value to meshless methods. The adopted NGF procedure was first presented in the work by Telles et al. [10] and is used to compute the fundamental solution for the MFS [9,12].

One class of the meshless methods that has been applied to a large range of problems is the local Petrov–Galerkin (MLPG) method presented in the work by Atluri and Zhu [13]. The MLPG is a truly meshless method, not requiring any type of mesh discretization like BEM, FEM or even meshless methods that use cell representations to compute the integral [1,14]. However, this flexibility in solving engineering problems can be computationally expensive in some cases. By using the MLPG to solve LEFM problems one needs to introduce several points near the crack tips, what may lead to a computationally expensive procedure.

The purpose of the present paper is to use an efficient iterative coupling procedure to solve LEFM problems. The problem domain is divided in sub-domains. Here, the NGF procedure is adopted for embedding a precise crack representation within the MFS idea while for standard elastic regular sub-domains the MLPG is adopted. This strategy permits to solve the principal problem in a decoupled manner without the necessity to introduce several near crack tip points to capture accurate stress intensity factors (SIF), as in the standard MLPG approach for fracture mechanics applications found in the works [5,15,16]. The adoption of MFS with the NGF approach to selectively represent the cracks of the

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http://dx.doi.org/10.1016/j.enganabound.2014.11.030
0955-7997/

problem alleviates the computer time burden found in traditional MLPG variations.

This text is organized as follows: first the governing equations and the numerical Green's function for crack problems are introduced; then we briefly present the MFS and the MLPG-1 method. The MLPG-1 is a variation of the MLPG, that uses the MLS weight function as test function in the local weak form [19]. The iterative coupling procedure is detailed as well as the choice of the optimized parameter is introduced. Finally two examples of problems solved by using the iterative coupling between the MFS-NGF and the MPLG-1 are shown to illustrate the degree of approximation found for the SIF.

2. Governing equations

For a two-dimensional linear elastic body, Ω, bounded by the boundary, Γ, the well-known Navier equation in terms of displacements u_i (generalized i directions displacements) can be written in the form

$$G\,u_{j,kk}+\frac{G}{\gamma}u_{k,kj}+b_j=0. \tag{1}$$

where G is the shear modulus, ν is the Poisson's ratio, $\gamma=1-2\nu$ and b_j is the body force components. The displacement u_i is solved from the Eq. (1) satisfying the boundary conditions:

$$\begin{aligned} &u_i=\overline{u}_i \quad \text{on } \Gamma_u \\ &p_i=\sigma_{ji}n_j=\overline{p}_i \quad \text{on } \Gamma_p \end{aligned} \tag{2}$$

In the above equation, $\overline{u}_i$ and $\overline{p}_i$ are the prescribed displacements and tractions on the boundary Γ_u and Γ_p, respectively. The external boundary of the body is $\Gamma=\Gamma_u\cup\Gamma_p$.

The Kelvin fundamental solutions for the second-order partial differential equations (1) $u_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})$ and $p_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})$ are given by [17]

$$u_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})=\frac{-1}{8\pi(1-\nu)G}\{(3-4\nu)\ln(r)\,\delta_{ij}+r_{,i}r_{,j}\} \tag{3}$$

$$p_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})=\frac{-1}{4\pi(1-\nu)r}\left\{[\gamma\delta_{ij}+2r_{,i}r_{,j}]\frac{\partial r}{\partial n}-\gamma(r_{,i}n_j-r_{,j}n_i)\right\} \tag{4}$$

where δ_{ij} is the Kronecker delta symbol, $u_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})$ and $p_{ij}^{\mathcal{K}}(\xi,\boldsymbol{\chi})$ correspond to the Kelvin fundamental displacement and traction in the j direction at the field point $\boldsymbol{\chi}$ on the infinite elastic medium subjected to a unit point load in i direction at the source point $\boldsymbol{\xi}$. Here $r=r(\boldsymbol{\xi},\boldsymbol{\chi})$ represents the euclidean distance between the load point $\boldsymbol{\xi}$ and the field point $\boldsymbol{\chi}$ whose derivatives $r_{,i}$ are taken with respect to the coordinates of $\boldsymbol{\chi}$. Eqs. (3) and (4) are valid for plane strain. For plane stress ν is replaced by $\overline{\nu}=\nu/(1+\nu)$.

Note that the LEFM problems are formulated based on the linear elasticity theory above. But due to the presence of cracks in the elastic medium, there will be surfaces sharing the same geometric position, this creates problems in the implementation of numerical methods. Difficulties like singularity of the system matrix or degeneration of the boundary integral equation [10] are expected to occur. So the MFS formulation needs special devices like the NGF procedure, discussed in the next section, to accommodate this.

3. Numerical Green's function (NGF)

The fundamental solution used in this work is the numerical Green's function [10,11]. The NGF is written in terms of a superposition of the Kelvin fundamental solution and a complementary part, which ensures that the final result is equivalent to an embedded crack unloaded within the infinite elastic medium subject to an unit applied load, given by

$$u_{ij}^{*}(\boldsymbol{\xi},\boldsymbol{\chi})=u_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})+u_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi}) \tag{5}$$

$$p_{ij}^{*}(\boldsymbol{\xi},\boldsymbol{\chi})=p_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})+p_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi}) \tag{6}$$

where $u_{ij}^{*}(\boldsymbol{\xi},\boldsymbol{\chi})$ and $p_{ij}^{*}(\boldsymbol{\xi},\boldsymbol{\chi})$ are the fundamental displacements and tractions in j direction at the field point $\boldsymbol{\chi}$ due to unit point load applied at the source point $\boldsymbol{\xi}$ in i direction ,respectively. The kernels $u_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})$ and $p_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})$ represent the known Kelvin's fundamental solution for the uncracked body already defined in Eqs. (3) and (4). Here, $u_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})$ and $p_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})$ stand for complementary components of the problem defined as an infinite space containing crack(s) of arbitrary geometry under applied distributed loads required to cancel the Kelvin's tractions as required in the original fundamental problem.

The NGF procedure presents a suitable feature for mesh-free methods, it introduces the existing crack surfaces without the need to include additional boundary condition points there for the problem representation. This is guaranteed by the traction-free crack representation simulated by the addition of the complementary solutions $u_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})$ and $p_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})$.

Analytical expressions for Eqs. (5) and (6) are limited to a few particular 2-D geometries. A general alternative to obtain the complementary solutions in a real variable numerical approach can be found in [10]. Consider $\boldsymbol{\chi}\notin\Gamma^f$ and using the Somigliana's identity, the complementary solutions can be defined in terms of the following boundary integral equations:

$$u_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})=\int_{\Gamma^-}p_{jk}^{\mathcal{K}}(\boldsymbol{\chi},\boldsymbol{\zeta})c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta})\,d\Gamma(\boldsymbol{\zeta}) \tag{7}$$

$$p_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})=\int_{\Gamma^-}P_{jk}^{\mathcal{K}}(\boldsymbol{\chi},\boldsymbol{\zeta})c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta})\,d\Gamma(\boldsymbol{\zeta}) \tag{8}$$

where $c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta})=u_{ik}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\zeta}^+)-u_{ik}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\zeta}^-)$ is the crack opening displacements of the Green's function. Here, Γ^+ and Γ^- stand for superior and inferior surfaces of the crack $\Gamma^f=\Gamma^+\cup\Gamma^-$ with $\boldsymbol{\zeta}\in\Gamma^-$, respectively. Also $P_{jk}^{\mathcal{K}}(\boldsymbol{\chi},\boldsymbol{\zeta})$ originates from the hypersingular formulation, the expression can be seen in [9].

The components $P_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})$ and $p_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\chi})$ are known, hence the crack opening displacements $c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta})$ need to be computed to produce the complementary components of displacement and traction (Eqs. (7) and (8)) and generate the fundamental numerical Green's function defined in Eqs. (5) and (6). Eqs. (7) and (8) can be solved numerically using a Gaussian quadrature [10]:

$$u_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})=\sum_{n=1}^{Npi}p_{jk}^{\mathcal{K}}(\boldsymbol{\chi},\boldsymbol{\zeta}_n)c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta}_n)|J|_nW_n \tag{9}$$

$$p_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\chi})=\sum_{n=1}^{Npi}P_{jk}^{\mathcal{K}}(\boldsymbol{\chi},\boldsymbol{\zeta}_n)c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta}_n)|J|_nW_n \tag{10}$$

where $|J|_n$ is the Jacobian of the transformation to the standard quadrature interval, $\boldsymbol{\zeta}_n$ and W_n are the corresponding point over the crack surface and weighting factor at the Gauss station n, respectively and Npi is the number of integration points. Prescribing traction boundary conditions $p_{ij}^{\mathcal{C}}(\boldsymbol{\xi},\boldsymbol{\zeta})=-p_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\boldsymbol{\zeta})$ along the crack surface and evaluating the limit of Eq. (8) as $\boldsymbol{\chi}\mapsto\Gamma^-$, the following hyper-singular boundary integral equation for unknowns $c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta})$ can be written:

$$\oint_{\Gamma^-}P_{jk}^{\mathcal{K}}(\overline{\boldsymbol{\zeta}},\boldsymbol{\zeta})c_{ik}(\boldsymbol{\xi},\boldsymbol{\zeta})\,d\Gamma(\boldsymbol{\zeta})=-p_{ij}^{\mathcal{K}}(\boldsymbol{\xi},\overline{\boldsymbol{\zeta}}) \tag{11}$$

where the symbol $\oint$ indicates Hadamard's finite part integral and $\overline{\boldsymbol{\zeta}}\in\Gamma^-$. The point collocation technique is adopted to solve Eq. (11); hence the following square system of equations, in matrix