Engineering Analysis with Boundary Elements 37 (2013) 1594–1601

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Solving an eigenvalue problem on a periodic domain using a radial basis function finite difference scheme

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ARTICLE INFO

Article history:
Received 18 February 2013
Accepted 28 August 2013
Available online 15 October 2013

Keywords:
Radial basis functions
Periodic boundary conditions
Eigenvalue problem
RBF-FD scheme

ABSTRACT

Local radial basis functions (RBFs) are becoming increasingly popular as an alternative to global RBFs, as the latter suffer from ill-conditioning. In this paper, a local meshless method based on RBFs in a finite-difference (FD) mode with better conditioned matrices has been developed for solving an eigenvalue problem with a periodic domain. Through numerical experiments, we examine the accuracy of the method as a result of variation in the number and layout of nodes in the domain and the effects of shape parameter, using various globally supported RBFs. The presented scheme has been validated on two different types of nodal arrangement, namely uniform and non-uniform node distributions. The results obtained from the method are found to be in good agreement with the benchmark analytical solutions. In addition, a higher-order RBF-FD scheme (which uses ideas from Hermite interpolation) is then proposed for solving the eigenvalue problem with a periodic domain. Tests show that both accuracy and convergence order can be improved dramatically by using higher-order RBF-FD formulae, which converge at a rate of $\mathcal{O}(h^{8.5})$ compared to the standard-order method which converges as $\mathcal{O}(h^{4.3})$ for uniformly distributed nodes with spacing h.

1. Introduction

There are many engineering problems which give rise to a set of partial differential equations (PDEs) along with some boundary conditions when modelled mathematically. These problems were traditionally solved using various mesh-based methods, including the finite element method (FEM), finite difference method (FDM) and finite volume method (FVM). These methods are popular and well-established, but suffer certain drawbacks due to their reliance on a mesh of elements connected in a predefined way. Generating the mesh can be computationally expensive, and sometimes it is necessary to re-mesh to achieve sufficient accuracy. There are also some problems, such as those involving large deformations, crack growth etc., which the mesh based methods cannot model well. Due in part to these limitations, there is growing interest in meshless methods for solving PDEs in engineering applications [1,2].

Meshless methods have a history traceable back to early collocation methods published in the 1930s for the purpose of computing excited electronic energy bands in metals [3,4]. A number of meshless methods have been introduced to date. Some of the well-known meshless methods are smooth particle hydrodynamics (SPH) method [5], diffuse approximate and diffuse element methods (DEMs) [6], element free Galerkin (EFG) method [7], reproducing kernel particle method (RKPM) [8], partition of unity method (PUM) [9], finite point method (FPM) [10], meshless local Petrov–Galerkin method (MLPG) [11], and meshfree weak-strong form methods [12,13]. In recent years, another group of meshless methods which are based on so-called RBFs have become attractive for solving PDEs (e.g., [14–18]). One of the main advantages of using meshless methods is that it is computationally easy to add or remove nodes from a pre-existing set of nodes, which is not the case for mesh-based methods, where the addition or removal of a point/element would lead to heavy remeshing, requiring significant computational resources.

The most commonly used globally supported RBFs (GSRBFs) in the literature for solving PDEs are multiquadrics (MQ), inverse multiquadrics (IMQ), polyharmonic splines (PS) and Gaussians [19]. MQ, IMQ and Gaussian RBFs include a shape parameter, whose numerical value can be varied to control the domain of influence of the basis function (for example, in the case of the Gaussian RBF, increasing the value of the shape parameter leads to flatter basis functions). However, these global RBFs produce dense matrices, which tend to become poorly conditioned as the number of collocation points increases. Their accuracy and efficiency has recently been shown to be poor for solving 3D parabolic PDEs in cases where the number of nodes is large [20], whereas local methods were found to be more efficient and accurate but displayed greater sensitivity to node position distributions and the choice of shape

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http://dx.doi.org/10.1016/j.enganabound.2013.08.018

parameter. There are currently several ways to overcome the disadvantages of using GSRBFs for solving PDEs, such as domain decomposition, preconditioning, fine tuning the shape parameters of MQs [14,21], and using compactly supported RBFs (CSRBFs) [22–24,17,18].

In 2003, Lee, Liu and Fan introduced a new truly meshless approximation strategy for solving PDEs, based on the local multiquadric and local inverse multiquadric approximations [25]. Their approximation function is constructed without requiring any geometric data other than the local configuration of nodes falling inside its influence domain. Šarler and Vertnik also formulated a local version of the classic RBF collocation method suitable for solving diffusion problems [26]. Among the recently proposed methods is the local RBF-FD. This method was found to give up spectral accuracy for a sparse, better-conditioned linear system and provide more flexibility for handling non-linearities when used to solve elliptic problems [27]. The sparseness and improved conditioning of the system matrices in the local RBF-FD method makes for quicker, more efficient computation than the full matrices used in the GSRBF method. The idea of using RBFs in a finite difference mode (RBF-FD) (which may be seen as a generalisation of the classical finite difference method to scattered node layouts) was introduced independently by Tolstykh and Shirobokov [28], and Wright and Fornberg [27], in the literature. Bayona et al. [29] have analytically studied the convergence behaviour of the local RBF method (using multiquadrics) as a function of shape parameter, number of nodes employed, and the nodal distance, finding an optimal shape parameter which is independent of nodal distance.

In this paper, the local RBF-FD is extended for solving an eigenvalue problem on a periodic domain. The method uses RBFs with global support, but uses the radial basis function in a finite difference mode, applying the RBFs inside a local support domain for each node, similarly to how finite difference schemes achieve sparsity. This reduces the number of so-called connections for each node, hence producing sparser and better-conditioned matrices than the global RBF methods, which produce increasingly ill-conditioned matrices as the number of nodes increases. The paper goes on to illustrate a higher-order RBF-FD formulation which employs ideas from Hermite interpolation to achieve better accuracy for a given size of support domain. In this work, numerical results are presented for several RBFs using regular and irregular node layouts, and compared to the analytical solutions of the problem.

The paper is structured as follows: Section 2 introduces radial basis function interpolation, Section 3 presents the formulation of the RBF-FD scheme and the higher-order RBF-FD scheme, and Section 4 gives the analytical solutions to our sample problem. In Section 5 we give the results of some computational experiments, and conclusions are drawn in Section 6.

2. Radial basis functions

For given data $\{\boldsymbol{x}_i, f_i\} \in \mathbb{R}^n \times \mathbb{R}, \quad 1 \le i \le N$ specifying values of a function $f : \mathbb{R}^n \to \mathbb{R}$ on a finite set of distinct centres $\{\boldsymbol{x}_i\}_{i=1}^N \in \mathbb{R}^n$, the approximation $F(\boldsymbol{x})$ to a function $f(\boldsymbol{x})$ can be written as

$$F(\boldsymbol{x}) = \sum_{j=1}^{N} \gamma_j \phi(\|\boldsymbol{x}-\boldsymbol{x}_j\|) + \beta, \tag{1}$$

where $\boldsymbol{x}$ and $\boldsymbol{x}_j$ are points in $\mathbb{R}^n$, the RBF is ϕ, the Euclidean norm on n-dimensional space is indicated by $\|\cdot\|$, and N is the total number of points. The coefficients γ_j and β may be found by setting

$$F(\boldsymbol{x}_i) = f(\boldsymbol{x}_i), \quad \text{for } i = 1, \ldots, N. \tag{2}$$

Table 1
Some commonly used globally supported RBFs.

Multiquadric
$\phi = (c^2 + r^2)^{1/2}$
Inverse multiquadric
$\phi = (c^2 + r^2)^{-1/2}$
Polyharmonic spline
$\phi = r^{2m} \log(r)$
Gaussian
$\phi = \exp(-r^2/c)$

and imposing that $\sum_{j=1}^{N} \gamma_j = 0$. We have augmented the interpolation with the constant β so that the interpolation is exact for constant functions, and this gives rise to a symmetric linear system of equations,

$$\begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{e} \\ \boldsymbol{e}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma} \\ \beta \end{bmatrix} = \begin{bmatrix} \boldsymbol{u} \\ 0 \end{bmatrix}, \tag{3}$$

with $\Phi_{i,j} = \phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|)$ for $i,j = 1, \ldots, N$, and $\boldsymbol{e}_i = 1$ for $i = 1, \ldots, N$. A low-degree augmenting polynomial (*e.g.*, of first degree) could instead have been chosen, and would ensure the exactness of the interpolation for linear and constant functions. However, this would lead to a non-linear system of equations for determining β and the n coefficients $\{\gamma_i\}_{i=1}^n$ of the polynomial of degree n, which would be computationally expensive.

In Table 1 we present some of the popular globally supported RBFs found in the literature. Note that c is a shape parameter which can take arbitrary values [30, p. 142]. In the case of the polyharmonic spline, m is a parameter taking positive integer values.

3. Meshless method formulation

We formulate our meshless method for the elliptic Helmholtz eigenvalue problem, which arises in many physical applications, in particular acoustic and electromagnetic waves [31]:

$$\nabla^2 u + \lambda^2 u = 0, \tag{4}$$

where ∇^2 is the Laplace operator. The problem is to find λ for which there exist non-null functions u (defined on n-dimensional Euclidean space $\mathbb{R}^n$) satisfying (4) and the given boundary conditions.

In physical applications, we usually have $n = 1$, 2 or 3. Here, we solve (4) in two dimensions, where it takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda^2 u = 0, \tag{5}$$

for a problem with periodic boundary conditions as illustrated in Fig. 1.

We make the boundary conditions periodic by requiring that

$$u(x, 0) = u(x, b),$$

and

$$u(0, y) = u(a, y),$$

where a and b are the lengths of the domain edges (see Fig. 1). Where so-called minimum image distances r are required between nodes, these are calculated as

$$r = \sqrt{\Delta_x^2 + \Delta_y^2}, \tag{6}$$

where the $\boldsymbol{x}-$component of the distance between the points at (x_i, y_i) and (x,y) is Δ_x and the $\boldsymbol{y}-$component is Δ_y, which are given by

$$\Delta_x = \min(\|x - x_i\|, a - \|x - x_i\|),$$