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Application of the Trefftz method, on the basis of Stroh formalism, to solve the inverse Cauchy problems of anisotropic elasticity in multiply connected domains

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ABSTRACT

In this paper, the Trefftz collocation method is applied to solve the inverse Cauchy problem of anisotropic elasticity, wherein both tractions as well as displacements are prescribed at a small part of the boundary of an arbitrary simply/multiply connected anisotropic elastic domain. The Stroh formalism is used to construct the Trefftz basis functions. Negative and positive power series are used together with conformal mapping to approximate the complex potentials of the Stroh formalism. For inverse problems where noise is present in the measured data, Tikhonov regularization is used together with the L-curve parameter selection method, in order to mitigate the inherent ill-posed nature of inverse problems. By several numerical examples, we show that this simple and elegant method can successfully solve inverse problems of anisotropic elasticity, with noisy measurements, in both simply and multiply connected domains.

1. Introduction

Computational modeling of solid/fluid mechanics, heat transfer, electromagnetics, and other physical, chemical & biological sciences have experienced an intense development in the past several decades. Tremendous efforts have been devoted to solving the so-called direct problems, where the boundary conditions are generally of Dirichlet, Neumann, or Robin type. Existence, uniqueness, and stability of the solutions have been established for many of these direct problems. Numerical methods, such as finite elements, boundary elements, finite volume, meshless methods, have been successfully developed and are available in many off-the shelf commercial softwares, see [1]. On the other hand, inverse problems, although being more difficult to tackle and being less studied, have equal, if not greater importance in the applications of engineering and sciences.

One of the many types of inverse problems is to identify sources or inaccessible boundary fields with over-specified measurements at only part of the boundary, i.e. the Cauchy problem. Take elasto-static solid mechanics as an example. Consider a domain of interest Ω, displacements $\overline{u}_i$ are prescribed at S_u, and tractions $\overline{t}_i$ are prescribed at S_t. If S_u and S_t form a complete division of $\partial\Omega$, i.e. $S_u \cup S_t = \partial\Omega, S_u \cap S_t = \varnothing$, then a direct problem is to be solved. Otherwise, if both the displacements $\overline{u}_i$ as well as tractions $\overline{t}_i$ are only prescribed at part of the boundary S_c, then an inverse Cauchy problem is to be solved.

In spite of its wide popularity, the finite element method is unsuitable for solving inverse problems. This is mainly because the symmetric Galerkin weak form prohibits one from prescribing both displacements as well as tractions at the same part of boundary. Therefore, in order to solve the inverse problem by FEM, one needs to iteratively solve a direct problem, and minimize the difference between the solution and measurement by adjusting the guessed boundary fields, see [2–4] for example. Recently, simple non-iterative methods have been under development for solving inverse problems without using the symmetric Galerkin weak-form: with global RBF as the trial function, collocation of the differential equation and boundary conditions leads to the global primal RBF collocation method [5,6]; with Kelvin's solutions as the trial function, collocation of the boundary conditions leads to the method of fundamental solutions, see [7,8]; with non-singular general solutions as the trial function, collocation of the boundary conditions leads to the boundary particle method [9–11]; with Trefftz trial functions, collocation of the boundary conditions leads to the Trefftz collocation method [12–15]. The common idea they

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share is that the collocation method is used to satisfy either the differential equations and/or the boundary conditions at discrete points. Collocation method is one of the most simple, efficient, and flexible methods, which allow both the tractions and displacements to be prescribed at the same location. Moreover, collocation method is also more suitable for inverse problems because measurements are most often made at discrete locations.

Among the various methods, the Trefftz method has shown extremely high efficiency and accuracy, provided that a relative complete trial function is used [14–18]. For a two-dimensional problem such as 2D Laplace equations and linear elasticity, the general solutions can mostly be expressed as analytic functions of complex variables. The completeness of the Trefftz trial functions therefore solely depends on how an analytic function should be approximated in a complex plane. Based on a detailed discussion in [14], a generalized Trefftz method is proposed to solve two-dimensional isotropic linear elasticity with arbitrarily shaped multiply connected domains. The later successful application of it in the direct numerical solution (DNS) of heterogeneous materials considering a large number of voids or inclusions [18–21], also demonstrated the ability of generalized Trefftz method in solving problems of 2D and 3D multiply connected domains.

In this paper, we combine and follow the work of [14] and [22–24] to apply the Trefftz method on the basis of Stroh Formalism to solve inverse problems of anisotropic elasticity, which was firstly dealt with in [25], in multiply connected domains. In Section 2, we introduce the Stroh formalism for two-dimensional anisotropic elasticity, with special attention being paid to how complex potentials $f_\alpha(z_\alpha)$ should be selected to construct the Trefftz trial functions. In Section 3, we give the detailed algorithm of Trefftz collocation method for inverse problems of anisotropic elasticity. Specifically, a simple regularization algorithm is given to increase the robustness of the algorithm when noise is considered. After that, several numerical examples are given in Section 4, to study the accuracy, convergence, and robustness of the proposed method. At last, some concluding remarks are made in Section 5.

2. Stroh formalism for anisotropic elasticity

Considering a linear elastic solid undergoing infinitesimal elasto-static deformations, the equations of linear and angular momentum balance, constitutive equations, and compatibility equations can be written as

$$\sigma_{ij,i}+\bar{f}_j=0,\quad \sigma_{ij}=\sigma_{ji}$$

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})\equiv u_{(i,j)} \tag{1}$$

where the Einstein summation convention on repeated indices is used.

This leads to the Navier's equations with displacements as primary variables, for a homogenous elastic body:

$$C_{ijkl}u_{k,li}+\bar{f}_j=0 \tag{2}$$

Here C_{ijkl} are the components of the fourth-order elasticity tensor for a homogenous solid.

For plane problems where body forces are negligible, the general solution of the Navier's equation (2) can be expressed through the Stroh Formalism. According to Ting's monograph [24], we have

$$\mathbf{u}=\sum_{q=1}^{4}\mathbf{a}_\alpha f_\alpha(z_\alpha) \tag{3}$$

$$z_\alpha=x_1+p_\alpha x_2 \tag{4}$$

$f_\alpha(z_\alpha)$ is an arbitrary analytic functions of z_α, and p_α and $\mathbf{a}_\alpha$ are the eigenvalues and eigenvectors of the following eigen-equation:

$$\{\boldsymbol{Q}+p(\boldsymbol{R}+\boldsymbol{R}^T)+p^2\boldsymbol{T}\}\boldsymbol{a}=0 \tag{5}$$

which is equivalent to

$$\begin{bmatrix} -\mathbf{T}^{-1}\mathbf{R}^{\mathbf{T}} & \mathbf{T}^{-1} \\ \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathbf{T}}-\mathbf{Q} & (-\mathbf{T}^{-1}\mathbf{R}^{\mathbf{T}})^{\mathbf{T}} \end{bmatrix}\begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\end{Bmatrix}=p\begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\end{Bmatrix} \tag{6}$$

For plane elasticity, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{T}$ are 2 by 2 real matrices given by

$$Q_{ik}=C_{i1k1}, R_{ik}=C_{i1k2}, T_{ik}=C_{i2k2} \tag{7}$$

Eq. (5) will give two pairs of conjugate solutions:

$$p_{\alpha+2}=\overline{p_\alpha},\ \mathbf{a}_{\alpha+2}=\overline{\mathbf{a}_\alpha},\ \mathbf{b}_{\alpha+2}=\overline{\mathbf{b}_\alpha},\quad \alpha=1,2 \tag{8}$$

Letting $f_{\alpha+2}=f_\alpha$, then Eq. (3) can be re-written as

$$\mathbf{u}=2\mathrm{Re}\sum_{\alpha=1}^{2}\mathbf{a}_\alpha f_\alpha(z_\alpha) \tag{9}$$

And corresponding stresses can be expressed as

$$\sigma_{i1}=-\Phi_{i,2}, \sigma_{i2}=\Phi_{i,1}$$

$$\mathbf{\Phi}=2\mathrm{Re}\sum_{\alpha=1}^{2}\mathbf{b}_\alpha f_\alpha(z_\alpha) \tag{10}$$

Now that general expressions for displacements and tractions have been worked out, the main issue is how the function $f_\alpha(z_\alpha)$ should be approximated for numerical implementation.

According to [13], when a simply connected domain is considered, it is reasonable to express the complex potentials with positive power series, representing modes of tension, shear, bending, etc.:

$$f_\alpha(z_\alpha)=\sum_{n=0}^{N}(iA_n^{\alpha o}+B_n^{\alpha o})(z_\alpha-z_\alpha^o)^n. \tag{11}$$

where $z_\alpha^o=x_1+p_\alpha x_2$ with (x_1,x_2) being the source point placed inside the domain.

For a doubly connected domain, we can locate the source point inside the cavity, and apply conformal mapping from z_α plane to

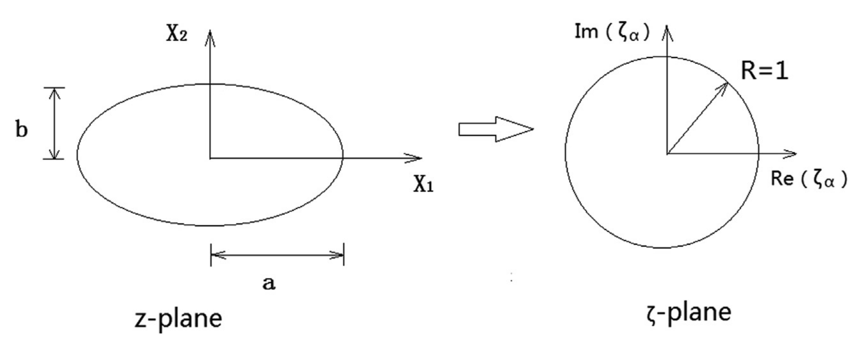

Fig. 1. Conformal mapping from an ellipse to a unit circle.