Engineering Analysis with Boundary Elements 43 (2014) 105–116

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

The complex variable fast multipole boundary element method for the analysis of strongly inhomogeneous media

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ARTICLE INFO

Article history:
Received 23 December 2013
Received in revised form
1 April 2014
Accepted 1 April 2014
Available online 25 April 2014

Keywords:
Inhomogeneous media
Harmonic and biharmonic problems
Complex variables
Singular and hypersingular boundary integral equations
Fast multipole method
Analytical integration
Effective moduli
Extreme value distributions

ABSTRACT

The paper aims to develop the efficient method tailored for accurate, robust and stable calculations of 2D local fields in strongly inhomogeneous materials with arbitrary interaction conditions on multiple contacts of structural blocks. The method is also to be of immediate use for solving homogenisation problems.

To reach the goal we employ: (i) special forms of the complex variable singular and hypersingular integral equations with the densities representing those physical quantities, which enter the contact conditions; (ii) circular-arc boundary elements (in addition to straight elements) for smooth approximation of smooth parts of the external boundaries and contacts; (iii) higher order approximations of densities, which account for arbitrary power asymptotics of physical fields near singular points (crack tips, corner points, common apexes of structural blocks); (iv) analytical recurrent evaluation of all influence coefficients; (v) analytical recurrent evaluation of all moments; (vi) the complex variable fast multipole method (CV FMM), for solving the resulting system of the complex variable boundary element method (CV BEM), with large number (up to million) of unknowns.

As a result, we obtain *free of numerical integration, higher-order CV fast multipole boundary element method* (*CV FM-BEM*) for a medium with multiple structural elements and multiple singular points. In the due course, we suggest the simplified starting quadrature formulae for singular boundary elements, the adjustment of the procedure for building the hierarchical tree and the proper choice of the key parameters of the developed CV FM-BEM: the number of elements in a leaf; the number of moments in the truncated Taylor expansions; the reasonable tolerance, when iteratively solving the system by the FMM.

Numerical examples illustrate the abilities of the method developed, as regard to local fields in strongly inhomogeneous structures with multiple singular points. The study of local fields shows application of the method to finding extreme distributions of stress intensity factors in a medium with many cracks, which may intersect. The homogenisation problem is solved, as well.

1. Introduction

Real materials being inhomogeneous at various structural scales, it is of value for many practical applications to get knowledge on local fields and average properties at a given level of structure. At a level with piece-wise homogeneous structure, the study unavoidably involves multiple interfaces of structural blocks, lines of their intersections and angular points. Commonly, the intersection lines and points are sources of strong field concentration, where unfavourable effects (fracture, corrosion, fatigue, energy loss, etc.) nucleate or accumulate. Therefore, from the physical point of view, accounting for multiple interfaces, lines and points of their intersections is crucial for proper modelling of physical processes. On the other hand, they strongly influence the accuracy of conventional numerical methods when finding local characteristics (potential, flux, stresses, strains, flux and/or stress intensity factors). Thus from the computational point of view, it is reasonable to use those numerical methods, which e asily account for field discontinuities, complicated interface conditions and asymptotic behaviour of fields near singular points.

In the case, when the structural blocks (including the matrix in composites) may be described by linear constitutive equations, the *direct* boundary integral equations (BIEs) (see, e.g. [1–14]) present

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http://dx.doi.org/10.1016/j.enganabound.2014.04.001
0955-7997/

the appropriate choice among the major classes of computational methods, including the finite element methods, finite difference methods, discrete element methods and particle methods. The direct BIEs are formulated in terms of physical quantities and they reduce the problem to considering the values of these quantities at interfaces. The direct BIEs become especially convenient for meeting non-trivial interface conditions, when the densities represent the very physical quantities, which enter the constitutive equations describing contact interactions.

In this paper, we consider 2D potential (harmonic) and elasticity (biharmonic) problems for strongly inhomogeneous media composed of piece-wise homogeneous structural blocks, which may have multiple cracks, pores and inclusions. Our aim is to develop the efficient general method tailored for accurate, robust and stable calculations of local fields in strongly inhomogeneous media. The method is also to be of immediate use for solving homogenisation problems.

Naturally, when considering plane problems, there is a possibility to use complex functions of the *complex variables* (*CV*) (see, e.g. [15–22]). Their analytical advantages over real variables, arising from holomorphicity of such functions in 2D elliptic problems, have been employed in numerous publications. Computational advantages of the CV are also significant; they have been summarised, for instance, in the introduction to the book [13], as concerns with solving the BIEs by conventional methods, and outlined in the papers [23–27], as regard to solving BIE by using the fast multipole method (FMM).

For problems, involving inclusions, the special forms of the CV equations have been employed for decades (see, e.g. [28–31]). The authors of these works focused on so-called perfect contact conditions, when both tractions and displacements are continuous through an interface. Commonly, these equations are of the indirect type what strongly complicates their extension to arbitrary interface conditions. The general forms of the CV equations, suggested in [32] for elasticity problems with arbitrary contact conditions, are still inconvenient for computations because, being singular, they contain as densities either the tangential derivative of the displacement discontinuities (DD) or the principal force, while the contact conditions involve the DD and the tractions. The latter quantities become the densities only when using hypersingular BIE.

The complex variable hypersingular integrals and complex variable hypersingular boundary integral equations (CV H-BIEs) were firstly introduced for elasticity problems [33,34]. Their theory was developed in [12,35–39] (a detailed review and complete theory may be found in [13]). Then the general CV H-BIE for elastic blocky systems with cracks, pores and inclusions, with arbitrary contact conditions, involving the displacement discontinuities and tractions, became available (e.g. [12,33,40–46]). Analogous CV equations for the potential problems have been derived later [47,48] following the same line.

Below, we use these CV BIEs as basic for our purpose. To solve them, we employ the special form of the CV BEM, which meets the requirements of computational efficiency (accuracy, robustness and stability). The accuracy is increased by using appropriate approximations for both the integration contour and the density functions. The smooth approximation of the smooth parts of the contour is reached by using circular-arc elements [12,39,43,49,50] (in addition to common straight elements for straight parts). As noted in [12], such elements provide the possibility to have continuous tangent at common edge points of neighbouring elements. What is also of significance, evaluation of the influence coefficients over such elements is easily performed by employing the recurrent analytical formulae, derived in [12,13,39,44,50] for densities approximated by basis functions from a very wide class. The class includes CV polynomials of an arbitrary order and the product of a power function with an arbitrary rational exponent and an arbitrary polynomial. The class is sufficient for accurate approximations of both smooth and singular behaviour of densities. Similar formulae in real variables look formidable, even for the lowest degrees of polynomials, and they hardly can be derived without using the CVs.

The mentioned recurrent analytical evaluation of the influence coefficients notably decreases the time expense, as compared with numerical integration. Numerical tests and experience (e.g. [12,13,37,39,41–43,46,49–51]), gained to the date, show that these forms of the CV BEM provide accurate and stable results. However, for strongly inhomogeneous media, the number of degrees of freedom (DOFs) becomes too large to solve the resulting algebraic system by the conventional exact or iterative methods. The problem of excessive growth of memory and time expense is overcome by employing the fast multipole method (FMM), suggested in [52,53].

The FMM employs two key ideas: (i) multipole expansions to account for combined influence of elements, which are far enough from a considered point, and (ii) a hierarchical tree with translations of moments to neighbouring levels of the tree. They reduce the number of operations, for iterative solving a system of order N, from N^3 to N. The FMM does not require keeping the whole matrix of the system. These features make it economic and robust. Its stability is reached by using appropriate pre-conditioners.

The computational gains, provided by the key elements of the FMM, have been used for solving problems with large number of DOFs (see, e.g. [14,23–27,54–61]). The detailed description of the FMM algorithm is given in the tutorial [24] and in the monograph [27], containing also examples, which may serve as benchmarks. The advantage of using the CV forms of the FMM is outlined in the papers [23–27]. Below we follow this line and perform all the calculations in the CV form.

Employing the FM-BEM involves integrals (multipole moments) additional to those defining the influence coefficients. In particular cases, they are evaluated analytically. Specifically, analytical formulae have been used to calculate moments for zero-order approximations of both contour and densities [23–27]. Another way to avoid numerical integrations is used when the boundaries have particular shapes of circles and/or straight segments [61]. It consists in looking for the solution in the form of complex Fourier series (for circles) and series of weighted Chebychev polynomials (for straight cracks). In this way the geometry is accounted for exactly, while the densities are approximated to high accuracy. In the general case of using higher-order approximations, the moments have been evaluated numerically (e.g. [62]). Meanwhile, to further reduce the time expense and integration errors, it is reasonable to use analytical rules rather than numerical integration. The possibility to obtain analytical formulae for higher-order approximations has been addressed in [63]. In this paper, we derive the needed recurrent analytical formulae for a wide class of higher-order approximations of both the contour and densities.

As a result, we obtain free of numerical integration, higher-order CV FM-BEM for a medium with multiple structural elements and multiple singular points. We employ: (i) the special forms of the CV singular and hypersingular BIE with the densities representing the physical quantities, which enter the contact conditions; (ii) circular-arc boundary elements (in addition to straight elements) for smooth representation of smooth parts of the external boundaries and contacts; (iii) higher order approximations of densities, which account for arbitrary power asymptotics of physical fields near singular points (crack tips, corner points, common apexes of structural blocks); (iv) analytical recurrent evaluation of all influence coefficients; (v) the CV FMM for solving the resulting system of the CV BEM with analytical recurrent evaluation of all moments.