Engineering Analysis with Boundary Elements 37 (2013) 1351–1366

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

# Numerical solution of Klein–Gordon and sine-Gordon equations by meshless method of lines

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ARTICLE INFO

*Article history:*
Received 14 January 2013
Accepted 5 July 2013
Available online 6 August 2013

*Keywords:*
Klein–Gordon equation
sine-Gordon equation
Meshless methods
Method of lines
Radial basis functions
Eigenvalue stability

ABSTRACT

We investigate numerical solution of the one dimensional nonlinear Klein–Gordon and two-dimensional sine-Gordon equations by meshless method of lines using radial basis functions. Results are compared with some earlier work showing the efficiency of the applied method. Salient feature of this method is that it does not require a mesh in the problem domain. Multiquadric and Gaussian are used as radial basis functions, which use a shape parameter. Choice of the shape parameter is still an open problem. We explore optimal value of the shape parameter without applying any extra treatment. For multiquadric, eigenvalue stability is studied without enforcing the boundary conditions whereas for Gaussian, the boundary conditions are enforced.



## 1. Introduction

In this work, we propose the meshless method of lines (MMOL) for the numerical solution of one-dimensional nonlinear Klein–Gordon (KG) and two-dimensional sine-Gordon (SG) equations. The method of lines (MOL) is a powerful numerical technique which approximates a time-dependent partial differential equation (PDE) in two steps. The spatial derivatives are approximated by finite difference (FD), finite element (FE) or any other algebraic approximations such as radial basis functions (RBFs), etc. Thus, the given PDE is reduced to a system of ordinary differential equations (ODEs). Any robust ODE solver can be invoked to solve the system of ODEs. A great deal of research has been devoted to MOL [1–4]. Several authors have used MOL for the numerical solution of time-dependent PDEs. These include nonlinear Burgers'-type equations [5], time-dependent nonlinear coupled PDEs [6], generalized Kuramoto–Sivashinsky equation [7], one-dimensional wave equation subject to an integral conservation condition [8], two-dimensional sine-Gordon equation [9], one-dimensional extended Boussinesq equation [4], nonlinear dispersive waves [3], nonlinear inverse heat conduction problems [10], Korteweg–de Vries (KdV) equation [2,11], parabolic inverse problem [12]. For a comprehensive list of applications see [8].

The meshless methods have an edge over the mesh-based numerical methods. In the present work, for the numerical solution of KG equation we use RBFs, Multiquadric (MQ) ($\phi(r)=\sqrt{r^2+c^2}$) and Gaussian (GA) ($\phi(r)=\exp(-cr^2)$), where $c$ is the shape parameter and plays a crucial role in the stability and convergence. Several authors have worked on finding the optimal shape parameter, these include Hardy [13], Foley [14], Carlson and Foley [15], Kansa and Carlson [16], Golberg et al. [17], Rippa [18], Kansa and Hon [19], Driscoll and Fornberg, [20], Fornberg et al. [21,22], Fornberg and Wright [23], and Larsson and Fornberg [24]. For more details, see [25] and references therein. Eigenvalue stability for explicit time integration schemes was investigated by Platte and Driscoll [26] and Sarra [27]. Platte and Driscoll showed that eigenvalue stability is possible in the absence of boundaries. They proved that differentiation matrices for conditionally positive definite RBFs are stable for periodic domains. They also showed that for Gaussian RBFs, special node distributions can achieve stability in 1-D and tensor-product non-periodic domains. Sarra investigated accuracy and eigenvalue stability of symmetric and asymmetric RBF collocation methods for hyperbolic initial boundary value problems in one and two dimensions. Accuracy and conditioning of RBFs interpolation were discussed by Schaback [28], Kansa and Hon [19], Fornberg et al. [29], and Platte and Driscoll [30]. Schaback proved the uncertainty relation between the error and the condition number of the interpolation matrices. He showed that the error and the condition number cannot be kept small at the same time. Kansa and Hon applied several techniques for improving the condition number of the coefficient matrix and the accuracy. Fornberg et al. and Platte and Driscoll used boundary treatment techniques to improve the accuracy. RBFs method was also used for problems like improved Boussinesq equation [31] and two-dimensional complex Ginzburg–Landau equation [32]. In the present work, we explore numerically how to choose

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http://dx.doi.org/10.1016/j.enganabound.2013.07.001

the optimum value of $c$ without doing any extra treatment. Eigenvalue Stability and convergence of the solution of nonlinear KG equation are discussed in the light of optimum shape parameter. The nonlinear KG equation is given by

$$\frac{\partial^2 u}{\partial t^2} + \alpha \nabla^2 u + \psi(u) = f(\mathbf{x}, t), \tag{1}$$

where $u = u(\mathbf{x}, t)$, $\mathbf{x} = (x, y)$, $\alpha$ is a known parameter and $\psi(u)$ is the nonlinear force. When $\psi(u) = \sin(u)$, Eq. (1) becomes the sine-Gordon (SG) equation [33]. We consider the following two cases of Eq. (1), namely, the one-dimensional nonlinear KG equation and the two-dimensional SG equation. The one-dimensional nonlinear KG equation is given by

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^2 = f(x, t), \tag{2}$$

with initial and boundary conditions

$$u(x, 0) = f_1(x), \quad u_t(x, 0) = f_2(x), \quad a \le x \le b, \tag{3}$$

$$u(a, t) = g_1(t), \quad u(b, t) = g_2(t), \quad t > 0. \tag{4}$$

In the above equations $u = u(x, t)$, and $\alpha, \beta, \gamma$ are known parameters.

The two-dimensional sine-Gordon equation is given by

$$u_{tt} = u_{xx} + u_{yy} - \sin(u) \tag{5}$$

with $u = u(x, y, t)$ in the region $\Omega = \{(x, y) : -a < x < a, -b < y < b\}$ for $t > 0$. We consider the Neumann boundary conditions

$$u_x = 0 \quad \text{for } x = -a \text{ and } x = a, \tag{6}$$

$$u_y = 0 \quad \text{for } y = -b \text{ and } y = b, \tag{7}$$

and the initial conditions

$$u(x, y, 0) = G_1(x, y) \quad \text{for } -a < x < a \text{ and } b < y < b, \tag{8}$$

$$u_t(x, y, 0) = G_2(x, y) \quad \text{for } -a < x < a \text{ and } b < y < b. \tag{9}$$

**Table 4**
Number of nodes, shape parameter and stable condition number by Störmer's method when $\delta t = 0.0001$ at $t$=0.1, corresponding to Example 1.

| $N$ | MQ | | GA | |
|---|---|---|---|---|
| | $c$ | Cnd number | $c$ | Cnd number |
| 11 | 0.75 | 1.04474E+010 | 3.13 | 4.54948E+011 |
| 21 | 0.36 | 2.06733E+011 | 24.24 | 2.92252E+012 |
| 31 | 0.24 | 7.25723E+011 | 65.12 | 3.74234E+012 |
| 41 | 0.18 | 1.45685E+012 | 121.73 | 5.84506E+012 |

**Table 1**
Comparison of results by calculating $L_\infty$ error norm, corresponding to Example 1, when $N$=11, $t$ = 0.0001.

| Method | $t$=0.01 | $t$=0.02 | $t$=0.1 | $t$=0.5 | $t$=1.0 |
|---|---|---|---|---|---|
| MQ-RK4 ($c$=1.2) | 4.74683E−007 | 8.91396E−007 | 1.90058E−006 | 8.16948E−005 | 3.18899E−004 |
| MQ-Störmer's ($c$=0.75) | 7.56392E−007 | 2.02243E−006 | 2.83297E−005 | 8.60692E−005 | 3.94758E−005 |
| GA-RK4 ($c$=2.19) | 4.86458E−007 | 9.41260E−007 | 3.86104E−006 | 1.26364E−004 | 7.08585E−004 |
| GA-Störmer's ($c$=3.13) | 5.35008E−007 | 1.13417E−006 | 6.80368E−006 | 4.62735E−005 | 4.20490E−004 |

**Table 2**
Condition (Cnd) number, spectral radius and the convergence rate for MQ with Störmer's method at $t$=0.1, corresponding to Example 1.

| *Spatial convergence rate, $c$=0.18 and $\delta t$ = 0.0001* | | | | | | |
|---|---|---|---|---|---|---|
| $N$ | Cnd number | $\rho$ | $L_2$ | $L_\infty$ | $L_2$ Rate | $L_\infty$ Rate |
| 11 | 8.41983E+003 | 8.71335E−006 | 2.63270E−003 | 7.18462E−003 | – | – |
| 21 | 6.56002E+006 | 3.67844E−005 | 4.06335E−004 | 1.50299E−003 | 2.69580 | 2.25708 |
| 31 | 3.36727E+009 | 8.42766E−005 | 6.94424E−005 | 2.32983E−004 | 4.35717 | 4.59780 |
| 41 | 1.45685E+012 | 1.51319E−004 | 1.28915E−005 | 4.46785E−005 | 5.85345 | 5.74062 |
| *Time convergence rate, $c$=0.75 and $N$=11* | | | | | | |
| $\delta t$ | Cnd number | $\rho$ | $L_2$ | $L_\infty$ | $L_2$ Rate | $L_\infty$ Rate |
| 0.0100 | 1.04474E+010 | 4.61836E−002 | 4.46205E−004 | 5.05989E−004 | – | – |
| 0.0010 | 1.04474E+010 | 4.61836E−004 | 4.74266E−005 | 5.54846E−005 | 0.97351 | 0.95997 |
| 0.0005 | 1.04474E+010 | 1.15459E−004 | 2.60844E−005 | 4.04354E−005 | 0.86251 | 0.45647 |
| 0.0001 | 1.04474E+010 | 4.61836E−006 | 1.12216E−005 | 2.83297E−005 | 0.52409 | 0.22107 |

**Table 3**
Condition number, spectral radius and the convergence rate for GA with Störmer's method at $t$=0.1, corresponding to Example 1.

| *Spatial convergence rate, $c$=130 and $\delta t$ = 0.0001* | | | | | | |
|---|---|---|---|---|---|---|
| $N$ | Cnd number | $\rho$ | $L_2$ | $L_\infty$ | $L_2$ Rate | $L_\infty$ Rate |
| 11 | 3.17937E+000 | 9.41632E−006 | 6.06798E−002 | 1.56509E−001 | – | – |
| 21 | 7.64635E+002 | 3.67005E−005 | 1.98187E−002 | 7.38432E−002 | 1.61436 | 1.08371 |
| 31 | 5.92834E+006 | 8.00058E−005 | 3.39633E−003 | 1.16738E−002 | 4.35038 | 4.54935 |
| 41 | 1.07986E+012 | 1.36533E−004 | 3.08964E−004 | 1.21925E−003 | 8.33290 | 7.85280 |
| *Time convergence rate, $c$=3.13 and $N$=11* | | | | | | |
| $\delta t$ | Cnd number | $\rho$ | $L_2$ | $L_\infty$ | $L_2$ Rate | $L_\infty$ Rate |
| 0.0250 | 4.54948E+011 | 1.06071E−001 | 1.26076E−003 | 1.35825E−003 | – | – |
| 0.0010 | 4.54948E+011 | 1.69714E−004 | 4.43198E−005 | 5.08359E−005 | 1.04013 | 1.02065 |
| 0.0005 | 4.54948E+011 | 4.24285E−005 | 2.21415E−005 | 2.56513E−005 | 1.00120 | 0.98682 |
| 0.0001 | 4.54948E+011 | 1.69714E−006 | 5.05455E−006 | 6.80368E−006 | 0.91781 | 0.82459 |