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Multiple elliptical inclusions of arbitrary orientation in composites ☆

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ABSTRACT

A volume integral equation method (VIEM) is introduced for the solution of elastostatic problems in an unbounded isotropic elastic solid containing multiple elliptical inclusions of arbitrary orientation subjected to uniform tensile stress at infinity. The inclusions are assumed to be long parallel elliptical cylinders composed of isotropic and anisotropic elastic material perfectly bonded to the isotropic matrix. The solid is assumed to be under plane strain on the plane normal to the cylinders. A detailed analysis of the stress field at the matrix–inclusion interface for square and hexagonal packing arrays is carried out, taking into account different values for the number, orientation angles and concentration of the elliptical inclusions. The accuracy and efficiency of the method are examined in comparison with results available in the literature.

1. Introduction

A number of analytical techniques are available for solving stress analysis of inclusion problems when the geometry of the inclusions is simple (i.e., cylindrical, spherical or ellipsoidal) and when they are well separated [1–4]. However, these approaches cannot be applied to more general problems where the inclusions are of arbitrary shape and their concentration is high. Thus the stress analysis of heterogeneous solids often requires the use of numerical techniques based on the finite element method (FEM) or boundary element method (BIEM). Unfortunately, both methods encounter limitations in dealing with problems involving infinite media or multiple inclusions. However, it has been demonstrated that a recently developed numerical method based on a volume integral formulation can overcome such difficulties in solving a large class of inclusion problems [5–13]. One advantage of the volume integral equation method (VIEM) over the boundary integral equation method (BIEM) is that it does not require the use of Green's functions for both the matrix and the inclusions. In addition, the VIEM is not sensitive to the geometry or concentration of the inclusions. Moreover, in contrast to the finite element method (FEM), where the full domain needs to be discretized, the VIEM requires discretization of the inclusions only.

The VIEM has been used successfully by Lee and his coworkers to solve a variety of stress analysis problems involving multiple inclusions or other geometrical imperfections embedded in an unbounded isotropic matrix. The specific problems considered include isotropic circular inclusions and cracks [5], fiber reinforced composites with partial debonding at the fiber matrix interface [6], circular inclusions and a void [7,8], and half plane problems with inclusions and a void [9,10]. As an extension, Dong et al. [11] applied the method developed by Lee and Mal [5] to investigate three dimensional problems involving one or two spherical inclusions and a single tetrahedral or hexahedral inclusion.

In the above mentioned papers [5–10], the inclusions were circles composed of isotropic or orthotropic materials. In [12,13] the inclusions were ellipses composed of isotropic or orthotropic materials, and they were assumed to have either square or hexagonal packing embedded in an unbounded isotropic elastic matrix subjected to remote uniaxial tension, in-plane shear or antiplane shear. The influences of the packing types, number of inclusions and concentration of the elliptical inclusions on the stresses at the matrix–inclusion interface were of particular interest. In those cases, however, the major axis of the elliptical inclusions was assumed to be parallel to the *x*-axis.

A micrograph of the cross section of a phosphate glass fiber/polymer composite is shown in Fig. 1 [14]. Fig. 1 indicates that the fibers are close to ellipses, and the major axis of the elliptical fibers is not aligned in any one direction. Therefore, in this paper, the effects of orientation angle on the elliptical inclusions are considered where the isotropic or orthotropic inclusions are assumed to have either square or hexagonal packing embedded in an unbounded isotropic elastic matrix under remote uniaxial tension. The elliptical inclusions are assumed to have different orientations relative to the loading direction. Of special interest here is the

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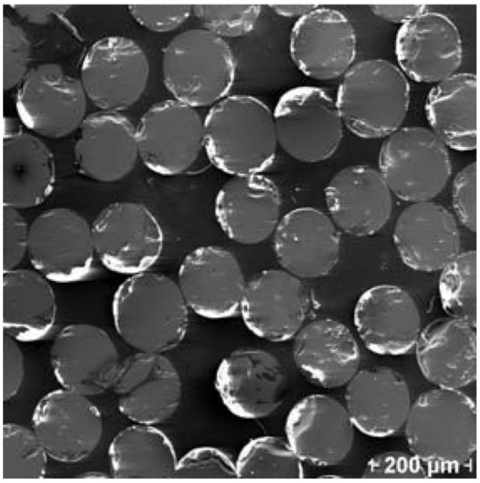

Fig. 1. Phosphate glass fiber/polymer composite cross section [14]. Used with permission from Journal of Materials Science: Materials in Medicine.

influence exerted by the packing types and orientation angles, as well as the number of elliptical inclusions and their fiber volume fractions on the stresses at the matrix–inclusion interface. It is demonstrated that the volume integral equation method is very accurate and effective in calculating the local stresses for these types of composites.

2. Volume integral equation method

The geometry of the general elastostatic problem considered here is shown in Fig. 2 where an unbounded isotropic elastic solid containing a number of isotropic or anisotropic inclusions of arbitrary shape are subjected to prescribed loading at infinity. The symbol $c_{ijkl}^{(1)}$ denotes the elastic stiffness tensor of the inclusion. $c_{ijkl}^{(2)}$ denotes the elastic stiffness tensor of the matrix material. The matrix is assumed to be homogeneous and isotropic so that $c_{ijkl}^{(2)}$ is a constant isotropic tensor, while $c_{ijkl}^{(1)}$ can be arbitrary, i.e., the inclusions may, in general, be inhomogeneous and anisotropic. The interfaces between the inclusions and the matrix are assumed to be perfectly bonded ensuring continuity conditions at the interface.

Let $u_m^o(\mathbf{x})$ denote the mth component of the displacement vector due to the applied remote stress at $\mathbf{x}$ in the absence of the inclusions. Also, let $u_m(\mathbf{x})$ denote the same in the presence of the inclusions. It has been shown in Mal and Knopoff [15] and Lee and Mal [5] that the elastostatic displacement in the composite satisfies the volume integral equation,

$$u_m(\mathbf{x}) = u_m^o(\mathbf{x}) - \int_R \delta c_{ijkl} g_{i,j}^m(\xi, \mathbf{x}) u_{k,l}(\xi) d\xi \tag{1}$$

where the integral is over the domain occupied by the inclusions, $\delta c_{ijkl} = c_{ijkl}^{(1)} - c_{ijkl}^{(2)}$, and $g_i^m(\xi, \mathbf{x})$ is the static Green's function (or Kelvin's solution) for the unbounded matrix material, i.e., $g_i^m(\xi, \mathbf{x})$ represents the ith component of the displacement at ξ due to unit concentrated force at $\mathbf{x}$ in the mth direction. In Eq. (1), the summation convention and comma notation have been used and the differentiations are with respect to ξ_i. It should also be pointed out that the integrand is non-zero only within the inclusions, since $\delta c_{ijkl} = 0$, outside the inclusions.

If $\mathbf{x} \in R$, then Eq. (1) is an integro-differential equation for the unknown displacement vector $\mathbf{u}(\mathbf{x})$. It can, therefore in principle, be determined through the solution of the equation. An algorithm for the solution of Eq. (1) was developed by Lee and Mal [5] by discretizing the inclusions using conventional finite elements.

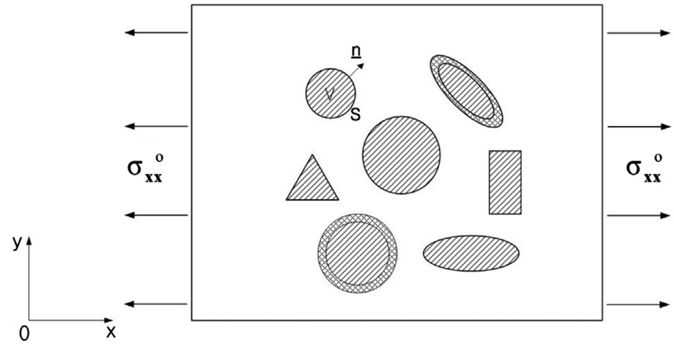

Fig. 2. Geometry of the general elastostatic problem.

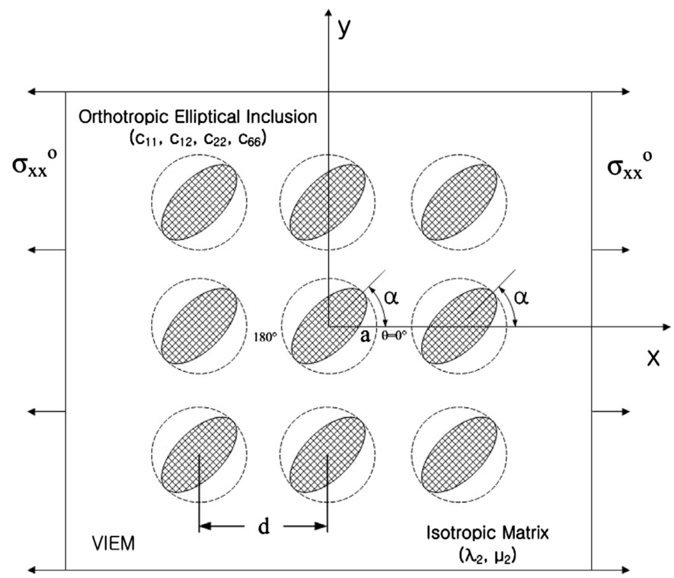

Fig. 3. Multiple orthotropic elliptical inclusions of arbitrary orientation in an unbounded isotropic matrix under uniform remote tensile loading.

Once $\mathbf{u}(\mathbf{x})$ within the inclusions is determined, the displacement field in the matrix can be calculated from Eq. (1) by evaluating the integral for $\mathbf{x} \notin R$. The stress field within and outside the inclusions can also be determined in a similar manner. The details of the numerical treatment of Eq. (1) for plane elastostatic problems can be found in [5]. Further explanation of the volume integral equation method for isotropic inclusions in an isotropic matrix can also be found in Section 4.3 "Volume Integral Equation Method" by Buryachenko [16].

3. Multiple inclusion problems

In this section we examine the effects of orientation angle on the elliptical fibers in composites. To illustrate this investigation, we consider plane strain problems for multiple isotropic or orthotropic elliptical cylindrical inclusions of arbitrary orientation in an unbounded isotropic matrix under uniform remote tensile loading, σ_{xx}^o, as shown in Fig. 3. For elliptical cylindrical inclusions, when the major axis coincides with the x axis at an inclined angle $\alpha = 0°$, the aspect ratio for each elliptical inclusion is assumed to be 0.5 [see Fig. 3]. α is defined as the orientation angle of each elliptical inclusion measured counterclockwise from the x-axis.

A detailed analysis of the stress field at the interface between the matrix and the central inclusion is carried out for square and hexagonal packing containing different numbers of isotropic or orthotropic inclusions in the unbounded matrix. The fiber volume