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A numerical study of Asian option with radial basis functions based finite differences method

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ABSTRACT

The purpose of this paper is to design and describe the valuation of Asian option by radial basis function approximation. A one state variable partial differential equation which characterizes the price of European type Asian option is discussed. The governing equation is discretized by the θ-method and the option price is approximated by radial basis function based finite difference method. Numerical experiments are performed with European option and Asian option and results are compared with theoretical and numerical results available in the literature. We show numerically that the scheme is second order accurate. Stability of the scheme is also discussed.

1. Introduction

Asian option is an option with the special feature that its payoff depends on arithmetic average of underlying asset price over its lifetime. Since no closed form solution is available to price Asian option, various strategies have been developed to examine it. To cite a few, Thompson [1] gave very constrict bounds, Geman and Yor [2] evaluated the Laplace transformation of Asian option but it was examined that for low volatility and short maturity its numerical inversion created difficulties as shown by Fu et al. [3]. Monte-Carlo simulation methods [4,5] work well for option pricing, but are computationally expensive.

Another track to look at Asian option is through solving partial differential equations in two space dimensions, which is prone to oscillatory solution. Ingersoll [6] showed that two dimensional floating strike Asian option can be transformed to one dimensional PDE. Rogers and Shi [8] provided a new transformation to model floating as well as fixed strike Asian option in one dimensional framework, and investigated new bound for it. Chen and Lyuu [9] successfully extended the concept of Rogers and Shi for general maturity. Several independent efforts have been made to price Asian option in recent years, see e.g. Zvan et al. [11], Vecer [12], Benhamou and Duguet [13], Zhang [14,15], Mudzimbabwe et al. [16] and references therein.

A new mesh free method for partial differential equations based on radial basis functions is currently undergoing intensive research. These methods aim to eliminate the structure of mesh and approximate the solution using a set of random points rather than points from grid discretization. A novel application of radial basis function to solve some interesting models can be found in [20–28]. Selection of optimal value of shape parameter is always difficult while working radial basis function based method. Considerable effort has been done in this direction and for details see [29–32] and reference therein.

Application of radial basis function based approach for solving PDEs arising in financial world has undergone active research, see e.g. [33–38] and references therein. To resolve issue related to stability and condition number of collocation matrix, Wright et al. [39–41] proposed radial basis function finite difference method; the idea is to use radial basis functions with a local collocation as in finite difference mode thereby reducing the number of nodes and hence producing a sparse matrix. In this strategy, it is expected that the choice of the shape parameter will not be a critical issue, as in case of global collocation method. In the present work, equation governing Asian option, derived by Alziary et al. [7] is discretized by using the well known θ-method on time interval and the option price is discretized by using radial basis function based scheme.

We will now describe the outline of the paper more precisely. In Section 2, we present the partial differential equation characterizing Asian option problem. The development of the scheme to solve the resulting equation is given in Sections 3. We study

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stability of the scheme in Section 4. Numerical results are presented in Section 5. Finally, conclusions are given in Section 6.

2. The model formulation for Asian option

The pricing of Asian options with arithmetically averaged strike price can be shown to satisfy a parabolic equation. It can be shown [18] that the two dimensional partial differential equation governing arithmetic Asian option is given as

$$\frac{\partial C}{\partial \tau}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial^2 S}+rS\frac{\partial C}{\partial S}+S\frac{\partial C}{\partial I}-rC=0 \tag{2.1}$$

$$C(S,I,T)=\max\left(\frac{I}{T}-K,0\right). \tag{2.2}$$

Here $S=S(\tau)$ denote the price of underlying asset, σ the volatility of underlying asset, r the risk free interest rate, which is fixed throughout the time period of interest, K the exercise price of the option, T the time of expiry and $I:=I(\tau)=\int_0^\tau S(\varphi)\,d\varphi$ denotes that the average price of the underlying asset in some time interval.

The value of Asian put option can easily be deduced from the Put-Call parity, now taking advantage of Put-Call parity [7,18], if the known part of average value is greater than strike price, i.e. $(K-(1/T)\int_0^\tau S(\varphi)\,d\varphi)\le 0$, then Geman and Yor [2] gave the approximation of call option as

$$C=\frac{S}{Tr}(1-e^{-r(T-\tau)})-e^{-r(T-\tau)}\left(K-\frac{1}{T}\int_0^\tau S(\varphi)\,d\varphi\right). \tag{2.3}$$

It is important to point out that the above problem is a two-dimensional PDE which is computationally expensive to solve. Thus taking advantage of homogeneous nature of $C(S,I,T)$ with respect to S and I, several attempts were made to reduce dimensional problem to make it easier from computational point of view. Rogers and Shi [8], Alziary et al. [7] have systematized new one dimensional partial differential equation for Asian option. By introducing new state variable,

$$\begin{cases} C=Su(y,\tau), \\ y=\dfrac{K-\frac{1}{T}\int_0^\tau S(\varphi)\,d\varphi}{S}, \end{cases} \tag{2.4}$$

the two dimensional PDE (2.1)–(2.2) can be reduced to one dimensional PDE as

$$\begin{aligned} &\frac{\partial u}{\partial \tau}+\frac{1}{2}\sigma^2 y^2\frac{\partial^2 u}{\partial y^2}+\left(-\frac{1}{T}-ry\right)\frac{\partial u}{\partial y}=0 \\ &u(y,T)=\max(-y,0). \end{aligned} \tag{2.5}$$

Now taking into account of the fact that value of option is known in the case $I\ge KT$ (i.e. $y\le 0$) and given by expression (2.3). By making the change of variables as in (2.4) we obtain

$$u=\frac{1}{Tr}(1-e^{-r(T-\tau)})-e^{-r(T-\tau)}y. \tag{2.6}$$

Hence we are required to solve the above PDE (2.5) only for $y\ge 0$ using (2.6) for the boundary condition at $y=0$. Therefore, we have the following PDE:

$$\frac{\partial u}{\partial \tau}+\frac{1}{2}\sigma^2 y^2\frac{\partial^2 u}{\partial y^2}+\left(-\frac{1}{T}-ry\right)\frac{\partial u}{\partial y}=0 \quad (y,\tau)\in(0,\infty)\times[0,T) \tag{2.7}$$

with terminal and boundary conditions

$$\begin{aligned} &u(y,T)=\max(-y,0), \quad y\in(0,\infty) \\ &u(0,\tau)=\frac{1}{Tr}(1-e^{-r(T-\tau)}), \quad \tau\in[0,T] \\ &\lim_{y\to\infty}u(y,\tau)=0, \quad \tau\in[0,T]. \end{aligned} \tag{2.8}$$

The boundary condition at infinity comes evidently from the definition of Asian call option, i.e. if strike become very large the option become worthless.

The resulting PDE given by (2.7) is backward in time and defined on positive real axis, so by introducing new state variable $x=e^{-y}$ and $t=T-\tau$, we get the final problem

$$\begin{aligned} &\frac{\partial u}{\partial t}=\frac{1}{2}\sigma^2x^2(\ln x)^2\frac{\partial^2 u}{\partial x^2}+\left[\left(\frac{1}{T}-r\ln x\right)x+\frac{\sigma^2}{2}x(\ln x)^2\right]\frac{\partial u}{\partial x}, \\ &(x,t)\in(0,1)\times(0,T] \end{aligned} \tag{2.9}$$

with initial and boundary conditions

$$\begin{aligned} &u(x,0)=0, \quad x\in(0,1) \\ &u(0,t)=0, \quad t\in[0,T] \\ &u(1,t)=\frac{1}{Tr}(1-e^{-rt}), \quad t\in[0,T]. \end{aligned} \tag{2.10}$$

We solve the resulting initial boundary value problem by using RBF based technique discussed in the next section. Once the solution $u(y,\tau)$ is obtained the price of the Asian option is determined by $C(S,I,\tau)=Su(y,\tau)$.

3. RBF approximation and time stepping

3.1. RBF-FD approximation of space operator

A function $\Phi:\mathbb{R}^d\to\mathbb{R}$ is called radial provided there exists a univariate function $\phi:[0,\infty)\to\mathbb{R}$ such that $\Phi(x)=\phi(r)$, where $r=\|x\|$ and $\|\cdot\|$ is some norm on $\mathbb{R}^d$. These functions can be broadly classified into two classes, infinitely smooth and piecewise smooth radial basis functions. The former include a shape parameter ϵ, and upon varying this parameter the radial function can vary sharp peak to very flat one. Classical choices of RBF are given in Table 1 with their order, where for any $x\in\mathbb{R}$, the symbol $\lceil x\rceil$ denotes as usual the smallest integer greater than or equal to x. The Gaussian and inverse multiquadric are positive definite functions whereas thin plate spline and multiquadric are conditionally positive definite functions of order $m>0$.

For completeness, a brief discussion of RBF based finite difference method is presented. To derive local RBF-FD approximation of any linear differential operator $\mathcal{L}:=d^k/dx^k$ of order k at a specific node point x_i, in the discretized domain $\Omega:=\{x_1,x_2,\dots,x_n\}$ containing n number of nodes, consider any subset Ω_i containing $n_i(\ll n)$ nodes in the neighborhood of x_i. In RBF-FD approach we are required to compute weights w_j such that

$$\mathcal{L}u(x_i)=\sum_{j=1}^{n_i}w_ju(x_j). \tag{3.1}$$

For each node $x_i\in\Omega$ we compute the weights w_j on each local support Ω_i. In traditional method, generally these nodes are equidistant and the weights are computed using classical polynomial interpolation. At the same time in radial basis function interpolation on randomly distributed nodes is used.

Table 1
Examples of radial basis functions and their order.

RBF	$\phi(r), r>0$	Order
Multiquadric (MQ)	$(1+(\epsilon r)^2)^\upsilon, \upsilon>0, \upsilon\notin\mathbb{N}$	$m=\lceil\upsilon\rceil$
Inverse multiquadric (IMQ)	$(1+(\epsilon r)^2)^\upsilon, \upsilon<0, \upsilon\notin\mathbb{N}$	$m=0$
Gaussian (GA)	$e^{-(\epsilon r)^2}$	$m=0$
Polyharmonic spline	$\begin{cases} r^\upsilon, \upsilon>0 & \text{if } \upsilon\in 2\mathbb{N}-1 \\ r^\upsilon\log(r) & \text{if } \upsilon\in 2\mathbb{N}\end{cases}$	$m=\begin{cases}\lceil\frac{\upsilon}{2}\rceil \\ \frac{\upsilon}{2}+1\end{cases}$