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PROMETHEE technique to select the best radial basis functions for solving the 2-dimensional heat equations based on Hermite interpolation

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ABSTRACT

In this work, we have decided to select the best radial basis functions for solving the 2-dimensional heat equations by applying the multiple criteria decision making (MCDM) techniques. Radial basis functions (RBFs) based on the Hermite interpolation have been utilized to approximate the solution of heat equation by using the collocation method. Seven RBFs, Gaussian (GA), Multiquadrics (MQ), Inverse multiquadrics (IMQ), Inverse quadrics (IQ), third power of Multiquadrics (MQ^3), Conical splines (CS) and Thin plate Splines (TPS), have been applied as basis functions as well. In addition, by choosing these functions as alternatives and calculating the error, condition number of interpolation matrix, RAM memory and CPU time, obtained by Maple software, as criteria, rating of cases with the help of PROMETHEE technique has been investigated. In the end, the best function has been selected according to the rankings.

1. Introduction

Radial basis functions (RBFs) interpolation is a technique for representing a function starting with data on scattered points. This technique first appears in the literature as a method for scattered data interpolation, and interest in this method exploded after the review of Franke [1], who found it to be the most impressive of the many methods he tested. Later, Kansa [2,3] proposed a scheme for the estimation of partial derivatives using RBFs. The main advantage of radial basis function methods is the meshless characteristic of them. The use of radial basis functions as a meshless method for the numerical solution of partial differential equations (PDEs) is based on the collocation method. These methods have recently received a great deal of attention from researchers [4–10].

Recently, RBFs methods were extended to solve various ordinary and partial differential equations including the high-order ordinary differential equations [11], the second-order parabolic equation with nonlocal boundary conditions [12,13], the nonlinear Fokker–Planck equation [14], regularized long wave (RLW) equation [15], Hirota–Satsuma coupled KdV equations [16], nonlinear integral equations [17,18], second-order hyperbolic telegraph equation [19], the solution of 2D biharmonic equations [20], the case of heat transfer equations [21] and so on [22–24].

A RBF $\Psi(\|\mathbf{x}-\mathbf{x}_i\|):\mathbb{R}^+\longrightarrow\mathbb{R}$ depends on the separation between a field point $\mathbf{x}\in\mathbb{R}^d$ and the data centers $\mathbf{x}_i$, for $i=1,2,\dots,N$, and N data points. The interpolants are classed as radial due to their spherical symmetry around centers $\mathbf{x}_i$, where $\|.\|$ is the Euclidean norm. Some of the infinitely smooth RBFs choices are listed in Table 1. The RBFs can be of various types, for example, Multiquadrics (MQ), Inverse multiquadrics (IMQ), Gaussian forms (GA) form, etc. In the cases of inverse quadratic, inverse multiquadric (IMQ) and Gaussian (GA), the coefficient matrix of RBFs interpolating is positive definite and, for multiquadric (MQ), it has one positive eigenvalue and the remaining ones are all negative [25].

One of the most powerful interpolation methods with analytic 2-dimensional test function is the RBFs method based on the Multiquadric (MQ) basis function

$$\psi(r)=\sqrt{r^2+c^2}, \tag{1}$$

suggested by Hardy [26], where $r=\|\mathbf{x}-\mathbf{x}_i\|$ and c is a free positive parameter, often referred to as the shape parameter, to be specified by the user. Madych and Nelson [27] showed that interpolation with MQ is exponentially convergent based on the reproducing kernel Hilbert space. Convergence property of the MQ

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Table 1
Some well-known functions that generate RBFs ($r = \|\mathcal{X} - \mathcal{X}_i\|_2 = r_i$), $c > 0$.

Name of functions	Definition
Multiquadrics (MQ)	$\sqrt{(cr)^2+1}$
Inverse multiquadrics (IMQ)	$\frac{1}{\sqrt{(cr)^2+1}}$
Inverse quadrics (IQ)	$\frac{1}{(cr)^2+1}$
Gaussian (GA)	$\exp(-(cr)^2)$
Third power of Multiquadrics (MQ[3])	$((cr)^2+1)^{3/2}$
Thin plate (polyharmonic) Splines (TPS)	$(-1)^{k+1}r^{2k}\log(r)$
Conical splines (CS)	r^{2k+1}

has also been proved by Buhmann [28,29]. Too large or too small shape parameter c in Eq. (1) makes the MQ too flat or too peaked. Despite many research works which are done to finding algorithms for selecting the optimum values of c [30–34], the optimal choice of shape parameter is an open problem which is still under intensive investigation.

The interested reader is referred to the recent books and paper by Buhmann [28,29] and Wendland [35] for more basic details about RBFs, compactly and globally supported and convergence rate of the radial basis functions.

Finding the basis function between several RBFs is an open important problem. In current work, we decide to select the best RBF for solving the 2-dimensional heat equations by taking convergence, condition number of interpolation matrix, CPU time and memory with a famous multiple criteria decision making (MCDM) method named PROMETHEE.

Today, various conditions under the influence of frequent and different factors and criteria have made the decisions complex. Consequently, we should apply the modern scientific methods to resolve them. Multiple criteria decision making (MCDM) problem is a well-known branch of decision theory. It has been found in real life decision situations [36–39]. In general, decision-making is the study of identifying and choosing alternatives based on the values and preferences of the decision-maker. Making a decision implies that some alternatives are to be considered, and that one chooses the alternative(s) that possibly best fits with the goals, objectives, desires and values of the problem. MCDM is a powerful tool used widely for evaluation and ranking problems containing multiple, usually conflicting criteria [40], as how it is in finding the best basis function in RBF methods.
A lot of researchers have devoted themselves to solve MCDM [41–46].

Among numerous methods of MCDM, the Preference Ranking Organization Method for Enrichment Evaluation (PROMETHEE) is significantly suitable for ranking applications [40]. PROMETHEE brings together flexibility and simplicity for the user [47] and is quite simple in conception and application compared to other methods for multicriteria analysis [48]. The PROMETHEE method and their applications have attracted much attention from academics and practitioners [49]. It is one of the best known and most widely applied outranking methods because it follows a transparent computational procedure and can be easily understood by actors and DMs [50].

This paper is arranged as follows: in Section 2, we describe the properties of radial basis functions. An approach based on radial basis functions to approximate the solution of linear operation by using the Hermite interpolation and collocation method is applied. In Section 3, the PROMETHEE methodology is described. We give computational results of numerical experiments with methods based on preceding sections, to support our theoretical discussion in Section 4. The conclusions are discussed in the final section.

2. Radial basis functions

2.1. Definition of radial basis functions

Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$ be the non-negative half-line and let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $\psi(0) \geq 0$. A radial basis function on $\mathbb{R}^d$ is a function of the form

$$\psi(\|\mathbf{x} - \mathbf{x}_i\|), \tag{2}$$

where $\mathbf{x}$, $\mathbf{x}_i \in \mathbb{R}^d$ and $\|.\|$ denotes the Euclidean distance between $\mathbf{x}$ and $\mathbf{x}_i$s. If one chooses N points $\{\mathbf{x}_i\}_{i=1}^N$ in $\mathbb{R}^d$ then by custom

$$s(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i \psi(\|\mathbf{x} - \mathbf{x}_i\|); \quad \lambda_i \in \mathbb{R} \tag{3}$$

is called a radial basis function as well [51].

The d-dimensional function $F(\mathbf{x})$, $F : \mathbb{R}^d \longrightarrow \mathbb{R}$, to be interpolated or approximated, can be represented by an RBFs as

$$F(\mathbf{x}) \approx F_N(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i \psi(\|\mathbf{x} - \mathbf{x}_i\|) + P_m(\mathbf{x}), \tag{4}$$

in the sense that

$$F(\mathbf{x}_j) = \sum_{i=1}^{N} \lambda_i \psi(\|\mathbf{x}_j - \mathbf{x}_i\|) + P_m(\mathbf{x}_j), \quad j = 1, 2, \ldots, N, \tag{5}$$

along with the constrains

$$\sum_{i=1}^{N} \lambda_i P_k(\mathbf{x}_i) = 0, \quad k = 1, 2, \ldots, m. \tag{6}$$

Here the numbers $\lambda_i, i = 1, 2, \ldots, N$, are real coefficients and Ψ is a radial basis function. The matrix formulation of the above interpolation problem can be written as $\Lambda \mathbf{X} = b$ with

$$\Lambda = \begin{bmatrix} \Psi & P_m \\ P_m^T & 0 \end{bmatrix},$$

$\mathbf{X}^T = (\lambda, \beta)$ and $b^T = (F, 0)$, where β are the coefficients of the polynomial.

The standard radial basis functions are categorized into two major classes [16]:

Class 1. *Infinitely smooth RBFs* [16,52]: These basis functions are infinitely differentiable and heavily depend on the shape parameter c e.g. Hardy multiquadric (MQ), Gaussian (GA), inverse multiquadric (IMQ), and inverse quadric (IQ). Here, we can get $P_m(\mathbf{x}) = 0$.

Class 2. *Infinitely smooth* (*except at centers*) *RBFs* [16,52]: The basis functions of this category are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions discussed in the Class 1. For example, thin plate spline and Conical splines [16].

2.2. RBFs interpolation based on Hermite approach

Assume we are given a domain $\Omega \subset \mathbb{R}^d$, and a linear operator of the form

$$L[u](\mathbf{x}, t) = H(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, t \in [0, T), \tag{7}$$

with initial and boundary conditions

$$I[u](\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad t = 0, \tag{8}$$

$$B[u](\mathbf{x}, t) = g(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \quad t \in [0, T). \tag{9}$$

It is possible to represent the solution u of the above time-dependent boundary value problem in terms of the following Hermite RBF (HRBF) interpolation:

$$u_N(\mathcal{X}) = \sum_{i=1}^{N_0} \lambda_i B^\star[\psi](\|\mathcal{X} - \xi_i\|) + \sum_{i=N_0+1}^{N_1} \lambda_i I^\star[\psi](\|\mathcal{X} - \xi_i\|)$$