Engineering Analysis with Boundary Elements 50 (2015) 97–108

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Direct use of radial basis interpolation functions for modelling source terms with the boundary element method

Carlos F. Loeffler [a,*], Átila L. Cruz [a], André Bulcão [b]

[a] PPGEM, Departamento de Engenharia Mecânica, Universidade Federal do Espírito Santo, Av. Fernando Ferrari, 514, Goiabeiras, 29075-910 Vitória, ES, Brasil
[b] PETROBRAS, CENPES—Centro de Pesquisa e Desenvolvimento Leopoldo A. Miguez de Mello, Av Horácio de Macedo—Cidade Universitária, 950, Ilha do Fundão, 21941-915 Rio de Janeiro, RJ, Brasil

ARTICLE INFO

Article history:
Received 6 December 2013
Received in revised form
6 June 2014
Accepted 26 July 2014
Available online 7 September 2014

Keywords:
Radial basis approximation
Poisson's problems
Boundary element method

ABSTRACT

In this paper a new technique is presented for transforming the domain integral related to the source term that characterizes the Poisson Equation, within the scope of the boundary element method, for two-dimensional problems. Similarly to the Dual Reciprocity Technique, the proposed scheme avoids domain discretization using primitive radial basis functions; however, it transforms the domain integral into a single boundary integral directly. The proposed procedure is simpler, more versatile and some useful and modern techniques related to radial basis function theory can be applied. Numerical tests show the accuracy of the proposed technique for a simple class of complete radial interpolation functions, pointing out the importance of internal poles and the potential of applying fitting interpolation schemes to minimize the computational storage, particularly considering more complex future approaches, in which a mass matrix may be generated. For the analysis of the accuracy and convergence of the proposed method, results are compared with those obtained using Dual Reciprocity, using known analytical solutions for reference.

1. Introduction

The boundary element method (BEM) has been shown to be an effective numerical technique for modeling many engineering problems. Considering problems governed by self-adjoint differential operators, numerical results of the BEM are of high accuracy, for which the properties of the fundamental solution play an important role.

In many cases, intrinsic BEM limitations have been overcome gradually through new strategies, such as the resolution of internal domain terms, for example, that which appears in the Poisson Equation. Despite the existence of many techniques that model this kind of problem, the Dual Reciprocity Technique (DRBEM) is still currently the most general option [1,2]. It was created for solving eigenvalue and dynamic problems, but its methodology has been successfully generalized and applied to other cases, especially to model the domain integrals that appear in time dependent cases, diffusive-advective problems and other heat and mass transfer problems [3,4].

Anticipating by some years the use of meshless approaches [5], the DRBEM also uses radial basis functions to interpolate the variable that comprises the kernel of the domain integrals [6]. However, the DRBEM was developed using radial functions with complete support, which are effective for interpolation [7,8] or fitting approximation. These classic functions provide satisfactory accuracy for application to moderately sized data sets, but difficulties can arise for larger data sets [9].

Considering the advances with the Compact Radial Basis Functions (CRBFs) [10,11], many criticisms have been made of the complete radial basis functions traditionally used in the DRBEM. CRBFs present advantages especially in cases in which the interpolation matrix is comprised of a large number of basis points and it can also be ill-conditioned.

In fact, computational tests have shown that many classic complete radial basis functions become inadequate with the DRBEM in certain applications. Some studies indicate lack of convergence when traditional complete radial functions are used in DRBEM in conjunction with iterative procedures [12]. It should be emphasized that the use of radial functions in this formulation differs from the simple interpolation procedure and also the techniques of solving differential equations, since it generates two primitive functions from the original interpolation function, forming auxiliary matrices that may produce additional harmful numerical effects.

* Corresponding author. Tel.: +55 27 33352669.
E-mail addresses: carlosloeffler@bol.com.br (C.F. Loeffler), atila.lc@hotmail.com (Á.L. Cruz), bulcao@petrobras.com.br (A. Bulcão).

http://dx.doi.org/10.1016/j.enganabound.2014.07.007
0955-7997/

Therefore, the objective of this paper is to present an alternative formulation, hereafter named DIBEM, to approximate the complete kernel of the domain integral related to the body force term, such as is done in simple interpolation. Similarly to the DRBEM, the standard approach of the BEM to the Laplace operator is maintained since it is assumed that it has satisfactory accuracy. The proposed technique is also simpler and more versatile, since any type of radial basis function may be used according to a simple scheme that avoids domain integrations, as it will be shown herein. In order to present preliminary results, only classic radial basis functions with complete support are tested.

2. Boundary integral equation

Consider a two dimensional domain $\Omega(X)$ constituted by a homogeneous and isotropic medium, in which a scalar potential $u(X)$ and a body force $p(X)$ act. The domain is limited by a boundary $\Gamma(X)$, where potential conditions are prescribed on $\Gamma_u(X)$ and normal derivatives of potential $q(X)$ on a complementary boundary $\Gamma_q(X)$. Taking an auxiliary function $u^*(\xi;X)$ and its normal derivative $q^*(\xi;X)$, where ξ is an arbitrary source point, it is possible to establish an equivalent inverse integral form related to Poisson's Equation, given by [13]:

$$c(\xi)u(\xi)+\int_\Gamma u(X)q^*(\xi;X)d\Gamma-\int_\Gamma q(X)u^*(\xi;X)d\Gamma=-\int_\Omega p(X)u^*(\xi;X)d\Omega \tag{1}$$

In Eq. (1), the value of the coefficient $c(\xi)$ depends not only on the positioning of the source point ξ with respect to the physical domain $\Omega(X)$ but also on the boundary smoothness at the same point [14].

3. Interpolation procedure

The aim is to resolve the integral term in the right hand side of Eq. (1) by interpolation, using radial basis functions in a similar approach as that used in the DRBEM procedure. Thus, the complete kernel of the domain integral is interpolated directly, according to the following expression:

$$p(X)u^*(\xi;X)=z(\xi;X)={}^{\xi}\alpha^j F^j(X^j;X) \tag{2}$$

For each source point ξ, the interpolation given by Eq. (2) is done by considering each of the base points X^j in relation to the domain points X, weighted by the coefficients ${}^{\xi}\alpha^j$. The quantity of basis points X^j must be equal to the known values of $z(\xi;X)$.

The interpolation functions F^j used belong to a class of radial functions, that is, the argument is composed by the Euclidian distance $r(X^j;X)$ between base points X^j and domain points X, hereafter called information points.

Non radial functions could also be used with success [15,16], but it is important to point out that in the DIBEM the kernel of the integral to be approximated is composed of the product of the fundamental solution and a function that describes the distribution of the body force on the domain. Therefore, for body forces with more general shapes, the radial basis functions become advantageous.

Regarding the interpolation functions, the following complete radial basis functions are tested:

$$F^j(X^j;X)=(r/\delta)(\text{simple radial}) \tag{3}$$

$$F^j(X^j;X)=(r/\delta)^3(\text{cubic radial}) \tag{4}$$

$$F^j(X^j;X)=(r/\delta)^2\ln(r/\delta)(\text{thin plate radial}) \tag{5}$$

The coefficient δ in these previous equations means the greatest distance between two nodal points.

Since the fundamental solution in the DIBEM is the kernel, the source point ξ must have different positions than the information points to avoid singularities. In fact, the body force is a known value, thus it can be taken at another position. Considering that linear boundary elements are used, the values of the body force are taken at the center.

Similarly to the DRBEM, the proposed method also uses a primitive function Ψ^j, such as

$$\int_\Omega z(\xi;X)d\Omega=\int_\Omega({}^{\xi}\alpha^j\psi^j_{,ii}(X))d\Omega=\int_\Gamma({}^{\xi}\alpha^j\psi^j_{,i}(X)n_i(X))d\Gamma={}^{\xi}\alpha^j\int_\Gamma\eta^j(X)d\Gamma \tag{6}$$

The numerical evaluation of the previous boundary integral is very simple. Using Eq. (6) and the well-known BEM procedure for the discretization, Eq. (1) may be rewritten as:

$$\begin{aligned}
&H_{11}u_1+\ldots H_{1n}u_n-G_{11}q_1-\ldots G_{1n}q_n={}^1\alpha_1N_1+{}^1\alpha_2N_2+..{}^1\alpha_nN_n\\
&H_{21}u_1+\ldots.H_{2n}u_n-G_{21}q_1-\ldots G_{2n}q_n={}^2\alpha_1N_1+{}^2\alpha_2N_2+..{}^2\alpha_nN_n\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&H_{n1}u_1+\ldots H_{nn}u_n-G_{n1}q_1-\ldots G_{nn}q_n={}^n\alpha_1N_1+{}^n\alpha_2N_2+..{}^n\alpha_nN_n
\end{aligned} \tag{7}$$

In a matrix form, for convenience:

$$[\mathbf{H}]\{\mathbf{u}\}-[\mathbf{G}]\{\mathbf{q}\}=[\mathbf{A}]\{\mathbf{N}\}=\{\mathbf{P}\} \tag{8}$$

In Eq. (8), the lines of matrix $\mathbf{A}$ are comprised by vectors ${}^{\xi}\boldsymbol{\alpha}$, which may be obtained from following the basic interpolation equation

$$[F][{}^{\xi}\boldsymbol{\alpha}]=[{}^{\xi}\boldsymbol{\Lambda}]u \tag{9}$$

Where the matrix ${}^{\xi}\boldsymbol{\Lambda}$ is composed of values of the fundamental solution. For each source point ξ, the right hand side of Eq. (9) may be rewritten as

$$[{}^{\xi}\boldsymbol{\Lambda}]u=[{}^{\xi}\boldsymbol{\Lambda}][F]\boldsymbol{\alpha} \tag{10}$$

Thus, the last two equations can be equaled, resulting in:

$$[{}^{\xi}\boldsymbol{\alpha}]=[\mathbf{F}]^{-1}[{}^{\xi}\boldsymbol{\Lambda}][\mathbf{F}]\boldsymbol{\alpha}=[\mathbf{F}]^{-1}[{}^{\xi}\boldsymbol{\Lambda}][\mathbf{p}] \tag{11}$$

For Poisson type problems, it is possible to optimize the computational operations, since the matrix ${}^{\xi}\boldsymbol{\Lambda}$ is diagonal. In both boundary formulations, DRBEM and DIBEM, an inverse must be computed once. However, in the first the construction of two interpolation matrices and additional products of them by **H** and **G**, both being all full matrices [2], are required. Thus, computational time is decreased for the DIBEM, which requires only the product of matrix ${}^{\xi}\boldsymbol{\Lambda}$ versus $\mathbf{F}^{-1}$ for each source point.

4. Internal basis points

As happens in the DRBEM, the distribution of the body force $p(X)$ inside the domain is not well approximated if the interpolation basis points are taken exclusively on the boundary. A first improvement to the accuracy of the results is simply to introduce basis points inside the domain, also named poles. Since the DIBEM interpolates directly all functions that compose the kernel of the domain integral, including the fundamental solution, a larger number of internal poles is required for better performance. However, unlike with the DRBEM, increasing the number of internal basis points does not alter the results, since the mathematical aim of the DIBEM is closely similar to an interpolation technique.

5. Least square curve-fitting

The similarity of the DIBEM to a direct interpolation procedure is advantageous, due to its simplicity. Thus, some auxiliary numerical