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Numerical solution of three-dimensional Laplacian problems using the multiple scale Trefftz method

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ABSTRACT

This paper proposes the numerical solution of three-dimensional Laplacian problems based on the multiple scale Trefftz method with the incorporation of the dynamical Jacobian-inverse free method. A numerical solution for three-dimensional Laplacian problems was approximated by superpositioning T-complete functions formulated from 18 independent functions satisfying the governing equation in the cylindrical coordinate system. To mitigate a severely ill-conditioned system of linear equations, this study adopted the newly developed multiple scale Trefftz method and the dynamical Jacobian-inverse free method. Numerical solutions were conducted for problems involving three-dimensional groundwater flow problems enclosed by a cuboid-type domain, a peanut-type domain, a sphere domain, and a cylindrical domain. The results revealed that the proposed method can obtain accurate numerical solutions for three-dimensional Laplacian problems, yielding a superior convergence in numerical stability to that of the conventional Trefftz method.

1. Introduction

Proposed by Trefftz in 1926 [1], the Trefftz method is a meshless numerical method for solving boundary value problems where approximate solutions are expressed as a linear combination of functions automatically satisfy governing equations. According to Kita and Kamiya [2], Trefftz methods are classified as either direct or indirect formulations. Unknown coefficients are determined by matching boundary conditions. Li et al. [3] provided a comprehensive comparison of the Trefftz method, collocation, and other boundary methods, concluding that the collocation Trefftz method (CTM) is the simplest algorithm and provides the most accurate solutions with optimal numerical stability. Applications of the Trefftz method in engineering problems, such as Laplace and biharmonic equations [4] and the two-dimensional boundary detection problem [5], have been reported.

The Trefftz method has recently become popular, since it is a numerical method for easily and rapidly solving the boundary value problems. Kita et al. [6] describe the application of the Trefftz method to the solution of three-dimensional Poisson equation. An inhomogeneous term containing the unknown function is approximated with a polynomial function in the Cartesian coordinates to determine the particular solution for the Poisson equation. Due to the complexity, most of the applications of the Trefftz method are still based on two-dimensional problems [7]. In the literature, to the best knowledge of the authors of this article, the formulation of Trefftz method based on the cylindrical coordinate system has not been found yet. In this study, we present the numerical solution of three-dimensional Laplacian problems by the collocation Trefftz method based on the cylindrical coordinate system.

In the present formulation, the unknown solution is approximated by superpositioning the T-complete functions satisfying the governing equation in the cylindrical coordinate system. The T-complete functions are composed of a set of linearly independent vectors. The basis for the T-complete functions includes 18 linearly independent functions. For the indirect Trefftz formulation, the solution is expressed as the linear combination of these basis functions. Because using conventional CTMs results in extremely ill-conditioned linear equation systems, particularly when solving three-dimensional Laplacian problems, the resulting numerical solutions may be unstable.

In order to obtain an accurate solution of the linear equations, special techniques [8], e.g., the Tikhonov regularization, the singular value decomposition, conditioning by a suitable preconditioner, and truncated singular value decomposition, may be required. Li et al. [9–11] have studied on the effective condition number for collocation methods and the stability analysis was made for the

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general collocation Trefftz methods, the method of fundamental solutions, and the collocation method using the radial basis functions.

Liu [12] has modified the Trefftz method, and refined it by incorporating a single characteristic length into the T-complete functions to reduce substantially the condition number of the resulting linear equation system. Moreover, Liu [13] proposed the multiple scale Trefftz method for solving the inverse Cauchy problem for the Laplace equation. Because applying the multiple-scale concept can significantly reduce condition numbers, the numerical solution for three-dimensional Laplacian problems was approximated based on the multiple-scale Trefftz method in this study.

In addition to the multiple scale Trefftz method, we adopted the general dynamical method proposed by Ku et al. [14]. The general dynamical method is based on the scalar homotopy method and demonstrates great numerical stability for solving linear algebraic equations, particularly for systems involving ill-conditioned problems. With the combination of the multiple scale Trefftz method and the dynamical Jacobian-inverse free method (DJIFM), solutions for extremely ill-conditioned systems of linear equations for three-dimensional Laplacian problems can be obtained.

The remainder of this paper is organized as follows: Section 2 describes the formulation of the Trefftz method for three-dimensional Laplacian problems based on cylindrical coordinate systems. Section 3.1 explains the derivation of the multiple-scale Trefftz method for solving three-dimensional Laplacian problems, and Section 3.2 demonstrates the incorporation of the DJIFM for solving the extremely ill-conditioned system of linear equations for three-dimensional Laplacian problems. In Section 4, the numerical solutions for three problems involving three-dimensional groundwater flow problems are addressed. Finally, conclusions are drawn in Section 5.

2. Trefftz formulation based on the cylindrical coordinate system

Considering a three-dimensional domain Ω enclosed by a boundary Γ, the Laplace governing equation is expressed as follows:

$$\nabla^2 u = 0 \text{ in } \Omega, \tag{1}$$

and

$$u = f \text{ on } \Gamma_D \tag{2}$$

$$u_n = \frac{\partial u}{\partial n} = \overline{q} \text{ on } \Gamma_N \tag{3}$$

where Ω denotes the object domain under consideration, n denotes the outward normal direction, Γ_D denotes the boundary where the Dirichlet boundary condition is given, and Γ_N denotes the boundary where the Neumann boundary condition is given. In this study, we adopted the cylindrical coordinate system, as shown in Fig. 1. The Laplace governing equation in the cylindrical coordinate system can be written as follows:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{4}$$

In the Trefftz method, the unknown solution is approximated by superpositioning the T-complete functions satisfying the governing equation, as shown in Eq. (4). The T-complete functions are composed of a set of linearly independent vectors:

$$\mathbf{N} = \{\overline{N_1}, \overline{N_2}, \overline{N_3}, \ldots, \overline{N_{18}}\} \tag{5}$$

The basis $\mathbf{N}$ for the T-complete functions includes 18 functions obtained from the separation of variables in the cylindrical coordinate system, which are listed in Appendix A. Therefore, for the indirect Trefftz formulation, one can say that the solution is written as the linear combination of these basis functions:

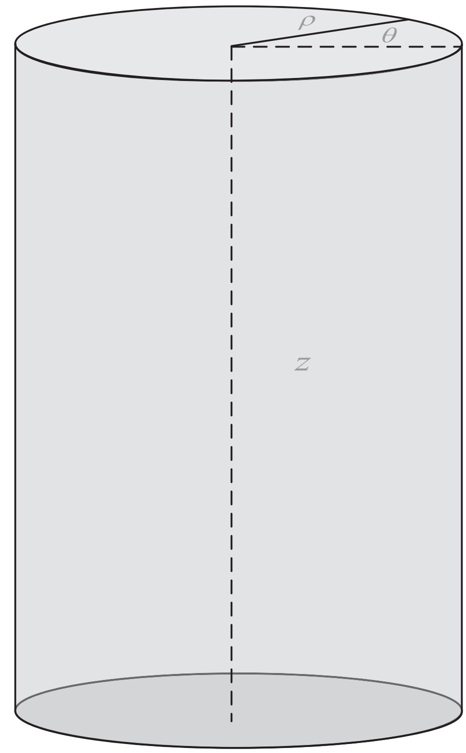

Fig. 1. The cylindrical coordinate system.

$$\begin{aligned} U = a + bz &+ \sum_{k=1}^{g} \left\{ \begin{array}{l} c_{1k}\cosh(kz)J_0(k\rho) + c_{2k}\sinh(kz)J_0(k\rho) \\ + c_{3k}\cos(kz)I_0(k\rho) + c_{4k}\sin(kz)I_0(k\rho) \end{array} \right. \\ &+ \sum_{\nu=1}^{h} \left\{ \begin{array}{l} d_{1k\nu}\cos(\nu\theta)\cosh(kz)J_\nu(k\rho) + d_{2k\nu}\sin(\nu\theta)\sinh(kz)J_\nu(k\rho) \\ + d_{3k\nu}\cos(\nu\theta)\sinh(kz)J_\nu(k\rho) + d_{4k\nu}\sin(\nu\theta)\cosh(kz)J_\nu(k\rho) \\ + d_{5k\nu}\cos(\nu\theta)\cos(kz)I_\nu(k\rho) + d_{6k\nu}\sin(\nu\theta)\sin(kz)I_\nu(k\rho) \\ + d_{7k\nu}\cos(\nu\theta)\sin(kz)I_\nu(k\rho) + d_{8k\nu}\sin(\nu\theta)\cos(kz)I_\nu(k\rho) \end{array} \right\} \Bigg\} \\ &+ \sum_{\nu=1}^{h} \{e_{1\nu}\rho^\nu\cos(\nu\theta) + e_{2\nu}\rho^\nu\sin(\nu\theta) + e_{3\nu}z\rho^\nu\cos(\nu\theta) + e_{4\nu}z\rho^\nu\sin(\nu\theta)\} \end{aligned} \tag{6}$$

where g and h are the order of the T-complete functions for approximating the solution. For determining the coefficients of a, b, c_{1k}, c_{2k},…, $e_{4\nu}$ in Eq. (6), we employ the collocation method. Eq. (6) can be discretized at a number of collocated points on the Dirichlet boundary. Then, we obtain a system of linear algebraic equations as follows:

$$\begin{bmatrix} 1 & z_1 & \cosh(kz_1)J_0(k\rho_1) & \cdots & z_1\rho_1^\nu\sin(\nu\theta_1) \\ 1 & z_2 & \cosh(kz_2)J_0(k\rho_2) & \cdots & z_2\rho_2^\nu\sin(\nu\theta_2) \\ 1 & z_3 & \cosh(kz_3)J_0(k\rho_3) & \cdots & z_3\rho_3^\nu\sin(\nu\theta_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_{aa} & \cosh(kz_{aa})J_0(k\rho_{aa}) & \cdots & z_{aa}\rho_{aa}^\nu\sin(\nu\theta_{aa}) \end{bmatrix} \begin{bmatrix} a \\ b \\ c_{1k} \\ \vdots \\ e_{4\nu} \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{aa} \end{bmatrix} \tag{7}$$

Eq. (7) can be written as follows:

$$\mathbf{Ay} = \mathbf{b}. \tag{8}$$

In Eq. (8), $\mathbf{A}$ is an $aa \times bb$ matrix, $\mathbf{y}$ is a $bb \times 1$ vector, and $\mathbf{b}$ is an $aa \times 1$ vector where aa is the number of the collocation points and bb is the number of T-complete functions. Considering the Neumann boundary condition, the collocation method can also be applied. Regarding a problem given in a Cartesian coordinate domain, the flux boundary where the Neumann boundary