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Generalized polyharmonic multiquadrics

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ABSTRACT

In this paper, we construct the two- and three-dimensional generalized polyharmonic multiquadrics (GPMQ) of order (K,L), which are the particular solution of the K-th order generalized multiquadrics (GMQ) associated with the L-th order polyharmonic operator for $L>0$. By observing the first few orders of the GPMQs, we construct methods of undetermined coefficients and determine the unknown coefficients by expanding the GPMQs into Laurent series. The derived GPMQs are hierarchically unique and infinitely differentiable. Then, the GPMQ definitions are extended for $L<0$ and the solutions are derived by similar methods. Both symbolic and floating-point implementations are performed for automatically obtaining the GPMQs of arbitrary orders, in which the former is explicitly provided and the later enables to implement numerical methods free from bookkeeping. The derived GPMQs are validated by numerical experiments, in which significant improvement on the accuracy can be observed.

1. Introduction

A radial basis function (RBF) is a real-valued function whose value depends only on the distance from a certain prescribed center. Traditionally, RBFs allow for scattered data to be easily used in computations. Among the various RBFs, the Gaussian function, Hardy's multiquadrics (MQ) [1], Duchon's augmented polyharmonic spline (APS) [2], and Wendland's compactly supported RBF (CSRBF) [3] are the more popular ones. A review by Franke [4] showed that the MQ performed better in the applications of the RBF interpolations (RBF-I).

In the early 1990s, Kansa [5,6] made the first attempt to use the MQ for approximating the solutions of partial differential equations (PDEs). The method, denoted as the global radial-basis-function collocation method (GRCM), is meshless, simple and has been used for a wide range of PDEs, such as solutions of Navier–Stokes equations [7], numerical wave tanks [8–10], natural convections in porous media [11] and solid–liquid phase change problems [12]. The advantage of the GRCM is its effectiveness in dealing with a complex domain.

Alternatively, Nardini and Brebbia [13] developed the dual reciprocity method (DRM) in which the RBFs were used for approximating the particular solution of the considered PDE and then the complementary solution was solved by boundary-type numerical methods, such as the boundary element method and the method of fundamental solutions (MFS). In the early development of DRM, the ad-hoc function, $1+r$, were exclusively used. In order to improve the accuracy of the computation, researchers applied the theory of radial basis functions (RBFs) to the DRM [14,15]. In 1996, Golberg et al. [16] showed the accuracy improvement of the MQ over other RBFs in the application of the dual reciprocity method of fundamental solutions (DRMFS).

When the MQ is applied in the RBF-I, GRCM and DRMFS, it results in ill-conditioned system matrix especially when high resolutions are considered. Recently, a localization procedure was proposed to transform the dense system matrices of the GRCM into sparse ones. Lee et al. [17] first proposed the local RBF collocation method (LRCM) based on the MQ. The LRCM had been applied to interdisciplinary fields, such as the solutions of diffusion problem [18], Darcy flow in porous media [19], macrosegregation phenomena [20] and others.

In addition to the MQ, the inverse MQ was also considered for problems with vanish far-field solutions in the original study of Hardy [1]. Furthermore, there were also researchers trying to improve the performance of the MQ by considering the generalized MQ (GMQ) [21–25]. In a recent review of Sarra and Kansa [21], they indicated that there seemed to be no particular advantage for the GMQ over the MQ. Alternatively, Sarra [26] considered the integrated MQ as a new RBF and addressed that the integrated MQ may produce significantly more accurate results over a wide range of shape parameters.

Furthermore, Chen et al. [27,28] used the particular solution of the MQ associated with the Laplace operator as a replacing RBF in the application of the GRCM and proposed the so-called method of approximate particular solutions. In their study, the particular

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solution was originally derived by Golberg et al. [16]. Analytical particular solutions of the MQ can also be found for the biharmonic operator [29] and the forth- and third- ordered polyharmonic operators respectively in two and three dimensions [30]. Recently, Tsai [31] derived the polyharmonic MQ (PMQ) which is the particular solution of the MQ associated with the polyharmonic operators and showed accuracy improvement of the PMQ over the MQ in the applications of DRMFS.

In this study, we will extend the study in [31] and derive the analytical particular solutions of the GMQ associated with the two- and three-dimensional polyharmonic operators. *Mathematica* codes will be implemented and provided for automatically deriving the generalized PMQs (GPMQs) of arbitrary orders. The GPMQs will be applied to all of the RBF-I, GRCM, DRMFS and LRCM.

This paper is organized as follows: the problem is mathematically modeled in Section 2. Then, the two- and three- dimensional GPMQs ($\boldsymbol{L>0}$) are derived in Sections 3 and 4, respectively. For $\boldsymbol{L\le 0}$, the GPMQs are given in Section 5. Some numerical experiments are carried out to validate the GPMQ in section 6 and the conclusions are drawn in Section 7.

2. Definition of the problem

In this study, we derive the two- and three-dimensional GPMQ $\psi_{K,L}^{(2D)}$ and $\psi_{K,L}^{(3D)}$ of order (K,L), which are governed by

$$\Delta_r^{(*)}\psi_{K,L}^{(*)}=\psi_{K,L-1}^{(*)} \tag{1}$$

with

$$\psi_{K,0}^{(*)}=\sqrt{c^2+r^2}^{\,2K+1}, \tag{2}$$

$$\Delta_r^{(2D)}=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right), \tag{3}$$

and

$$\Delta_r^{(3D)}=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right). \tag{4}$$

In Eqs. (1), (2) and the following, $(*)$ stands for $(2D)$ or $(3D)$ and K and L are arbitrary integers. If we make the following change of variables

$$r=cR, \tag{5}$$

$$\psi_{K,L}^{(*)}=c^{2L+2K+1}\Psi_{K,L}^{(*)}. \tag{6}$$

Eqs. (1) and (2) respectively become

$$\Delta_R^{(*)}\Psi_{K,L}^{(*)}=\Psi_{K,L-1}^{(*)}, \tag{7}$$

$$\Psi_{K,0}^{(*)}=\sqrt{1+R^2}^{\,2K+1}. \tag{8}$$

Using Eq. (8), Eq. (7) can be rewritten as

$$\left(\Delta_R^{(*)}\right)^L\Psi_{K,L}^{(*)}=\sqrt{1+R^2}^{\,2K+1} \tag{9}$$

for $L>0$. It is clear that $\Psi_{K,0}^{(*)}\to R^{2K+1}$ as $R\to\infty$ and therefore it is bounded only for $K\le -1$.

It can be noticed that the solutions of Eqs. (7) and (8) are also the solutions of Eq. (9). However, the converse is not always true since the homogeneous solutions of Eq. (9) can be arbitrarily chosen. In the next two sections, a specific choice will be made to ensure the solutions of Eq. (9) are also the solutions of Eqs. (7) and (8). Furthermore, the definition of GPMQs (7) can be extended to the cases of $L\le 0$ as

$$\left(\Delta_R^{(*)}\right)^{-L}\sqrt{1+R^2}^{\,2K+1}=\Psi_{K,L}^{(*)}. \tag{10}$$

Finally, the GPMQs governed by Eqs. (1) and (2) can be obtained by using Eq. (6).

3. Two-dimensional GPMQ for $L>0$

It is clear that the solution of Eq. (9) can be found by straight integrations. For the two-dimensional case, it can be expressed as follows:

$$\Psi_{K,L}^{(2D)}=\left(\mathbf{I}R^{-1}\mathbf{I}R\right)^L\sqrt{1+R^2}^{\,2K+1}, \tag{11}$$

where the integral operator $\mathbf{I}$ is defined as $\int dR$. When observing the first few orders of the GPMQs obtained by the symbolic software, *Mathematica* [32], we can find that

$$\Psi_{K,L}^{(2D)}=\sum_{j=0}^{M_{K,L}}\Phi_{K,L,j}^{(2D)}+\sum_{j=0}^{L-1}B_{K,L,j}^{(2D)}R^{2j}\ln\left(\frac{1+\sqrt{1+R^2}}{2}\right)+\sum_{j=0}^{L-1}C_{K,L,j}^{(2D)}R^{2j}+\sum_{j=0}^{L-1}D_{K,L,j}^{(2D)}R^{2j}\ln R \tag{12}$$

with

$$\Phi_{K,L,j}^{(2D)}=\begin{cases}A_{K,L,j}^{(2D)}R^{2j}\sqrt{1+R^2}\ \text{for } K\ge -2L\\ A_{K,L,j}^{(2D)}R^{2j}\sqrt{1+R^2}^{\,2K+4L+1}\ \text{for } K<-2L.\end{cases} \tag{13}$$

In Eq. (13), we have used the following definition:

$$M_{K,L}=\begin{cases}\text{Max}\{L+K,L-2\}\ \text{for } K\ge -2L\\ -K-L-2\ \text{for } K<-2L.\end{cases} \tag{14}$$

In Eq. (12), the first two series form the particular solutions with undetermined coefficients $A_{K,L,j}^{(2D)}$ and $B_{K,L,j}^{(2D)}$ and the last two series give the homogeneous solutions with arbitrary coefficients $C_{K,L,j}^{(2D)}$ and $D_{K,L,j}^{(2D)}$. To make $\Psi_{K,L}^{(2D)}$ infinitely differentiable, we obviously need

$$D_{K,L,j}^{(2D)}=0. \tag{15}$$

Then Eq. (12) becomes

$$\Psi_{K,L}^{(2D)}=\sum_{j=0}^{M_{K,L}}\Phi_{K,L,j}^{(2D)}+\sum_{j=0}^{L-1}B_{K,L,j}^{(2D)}R^{2j}\ln\left(\frac{1+\sqrt{1+R^2}}{2}\right)+\sum_{j=0}^{L-1}C_{K,L,j}^{(2D)}R^{2j}. \tag{16}$$

In Eq. (16), the coefficients $A_{K,L,j}^{(2D)}$ and $B_{K,L,j}^{(2D)}$ should be determined such that the GPMQ $\Psi_{K,L}^{(2D)}$ satisfies Eq. (9) and the arbitrary coefficients $C_{K,L,j}^{(2D)}$ should be chosen to ensure the GPMQ $\Psi_{K,L}^{(2D)}$ unique and also satisfying the hierarchical relation (7) and (8). Therefore, we expand Eq. (16) into the Maclaurin series as

$$\Psi_{K,L}^{(2D)}=\sum_{i=0}^{\infty}\sum_{j=0}^{\text{Min}\{M_{K,L},i\}}P_{K,L,i,j}^{(2D)}R^{2i}+\sum_{i=0}^{\infty}\sum_{j=0}^{\text{Min}\{L-1,i-1\}}Q_{K,L,i,j}^{(2D)}R^{2i}+\sum_{i=0}^{L-1}C_{K,L,i}^{(2D)}R^{2i} \tag{17}$$

with

$$P_{K,L,i,j}^{(2D)}=\begin{cases}\dfrac{(-1)^{i-j-1}(2i-2j-3)!!}{(i-j)!2^{i-j}}A_{K,L,j}^{(2D)}\ \text{for } K\ge -2L\\ \dfrac{(-1)^{i-j-K-2L-1}(2K+4L+1)!!(2i-2j-2K-4L-3)!!}{(i-j)!2^{i-j}}A_{K,L,j}^{(2D)}\ \text{for } K<-2L\end{cases} \tag{18}$$