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Higher order multipoint method – from Collatz to meshless FDM

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ABSTRACT

The higher order multipoint meshless method for boundary value problems is considered in this paper. The new method relies on the Collatz multipoint concept and the meshless FDM.

The main idea of multipoint technique is based on raising the local approximation order of a searched function by assuming additional degrees of freedom *at each node of stencil*, e.g. by including a combination of nodal values of the right-hand side of considered differential equation. Thus, the FD formula takes into account a combination of unknown function values which is equal a combination of additional d.o.f., e.g. right-hand side of PDE. In this way one may generate higher order FD operators without any additional unknowns using the same set of nodes in stencil as in the non-multipoint case.

Essential modifications and extensions of the old classical multipoint formulation have been introduced for the purpose of this research. New fully automatic multipoint meshless FDM uses the moving weighted least squares approximation instead of the interpolation proposed by Collatz, and is based on arbitrarily distributed cloud of nodes. Moreover, besides the local formulation, also various global formulations of b.v. problems are possible.

Several numerical benchmark problems analyzed illustrate the effectiveness of the proposed approach.

1. Introduction

Various ways exist in modern numerical analysis to improve the solution precision of boundary value (b.v.) problems. There are two main directions of such improvement. The first one is based on the mesh density increase, preferably using an adaptive (*h*-type) solution approach. The second option is provided by rising the approximation order (*p*-type). It may be raised in several ways. In the finite difference method (FDM) the oldest one is called the deferred correction technique [1]. It uses FD stars (stencils) with the increased number of nodes. Another way uses the same simple stars, but with additional generalized degrees of freedom (d.o.f.) introduced there [2]. The most recent concepts [3] are based either on evaluation of the higher order derivatives and using them as “correction terms” in the FDM equations [4], or on the modified multipoint approach [5] considered in this paper.

The new approach is based on the meshless finite difference method (MFDM) [2] using arbitrarily irregular grids as well as various formulations of b.v. problem. The multipoint MFDM provides raising the order of approximation by introducing additional degrees of freedom at the star nodes, taking into account a combination of searched function values together with combination of the right hand side values of the considered equation (specific approach) or other chosen operator (general approach). In this way one may generate higher order FD operators using the same set of nodes in a meshless FD (MFD) star (stencil, Fig. 1a) as in the non multipoint case. This increase the computational effectiveness in the *specific* multipoint version (Fig. 1b), where the higher order (HO) approach is obtained without any additional unknowns. The *general* version (Fig. 1c) of multipoint method instead provides general HO solution approach for all types of b.v. problems and various physical processes being modeled on the same geometric domain due to relations between unknown function and its derivatives.

The basic concept of the multipoint solution approach was introduced long time ago by Lothar Collatz [6] as improvement of the FDM. In Section 2 of this paper the main idea of the Collatz multipoint method is presented, whereas in Section 3 the features of multipoint concept are summarized. In his book [6] Collatz has described the essence of HO multipoint concept and has calculated multipoint FD formulas for several differential operators only due to computational difficulties with hand calculations. This basic multipoint approach has been reformulated by the authors and extended to the fully automatic multipoint meshless finite difference method which is discussed in Section 4. In comparison with the Collatz

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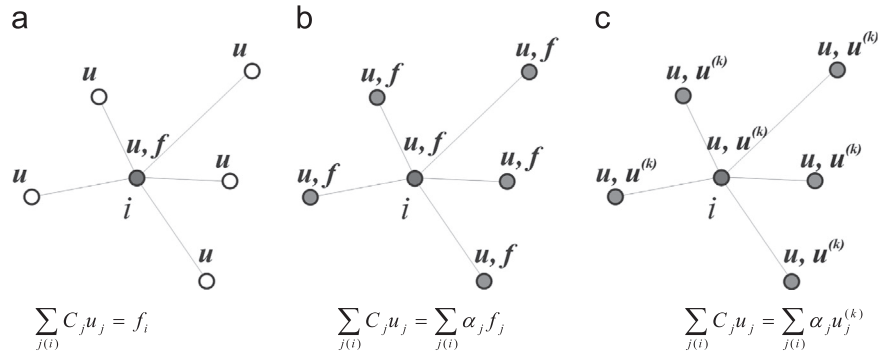

Fig. 1. MFD star used in (a) meshless FDM; (b) specific multipoint MFDM and (c) general multipoint MFDM.

approach, the following basic modifications have been included in the new method:

- the various global (weak) and global-local formulations of b.v. problems may be applied besides the local (strong) one;
- cloud of arbitrarily distributed nodes may be used instead of regular meshes; and
- the moving weighted least squares (MWLS) approximation technique [2,7] is used instead of the interpolation one.

The paper is illustrated with the selected results of benchmark tests (Section 5). Eventually, benefits of the proposed approach are highlighted in the final remarks (Section 6).

Only the main concept of the new multipoint MFDM approach is described here, having the details of particular problems to our other papers ([8] and articles that follows).

2. Interpolation based multipoint approach

As mentioned before, the main idea of the Collatz multipoint approach [6] relies on an improvement of the standard FDM due to introducing into the FD operator values of the right hand side of the considered differential equation (the specific approach) or the unknown function derivatives (the general approach) defined at nodes of the FD star (Fig. 1).

Thus, in the multipoint method two approaches are distinguished namely the specific and the general ones. Both of them provide higher order FD operators. The specific approach is more simple, but it may be applied only to linear boundary value problems. The general multipoint approach is more complex, but it may be used in all types of b.v. problem formulation, linear and non-linear ones.

2.1. Specific case

The specific approach is simpler than the general one, allows to provide HO solution without any additional unknowns, but its application is more restricted. Besides the d.o.f. assigned to unknown function values it also uses known values of a given differential operator (e.g. right hand side of ODE or PDE).

Consider the locally formulated boundary value problem

$$Lu = u^{(n)} + \sum_{q=0}^{n-1} l_q u^{(q)} = f(P), \quad u = u(P), \quad P \in \Omega, \tag{1}$$

with relevant boundary conditions

$$Gu = g(P), \qquad P \in \partial\Omega.$$

When the nodes in the domain Ω are introduced, one may select appropriate FD stars and discretize the given operator. Using an auxiliary function Φ summed over all nodes j ($j=1,2,\ldots m$) of the FD star with the central node i, the basic multipoint formula is assumed as follows:

$$\Phi_i = \sum_{j(i)} c_j u_j - \sum_{j(i)} \alpha_j f_j, \tag{2}$$

where $c_j = c_{j(i)}$, $\alpha_j = \alpha_{j(i)}$, and

$$\sum_{j(i)} c_j u_j = \sum_{j(i)} \alpha_j f_j \quad \text{for} \quad \Phi_i = 0. \tag{3}$$

To obtain this formula, one expands u_j and the right hand side of the considered equation $f_j = L\,u_j$ into the truncated (by including higher order terms and neglecting Δ_{0j} and Δ_j) Taylor series in respect of the stencil central node i

$$u_j = \bar{u}_j + \cancel{\Delta_{0j}}, \quad f_j = \bar{f}_j + \cancel{\Delta_j}, \qquad j = 1,2,\ldots m. \tag{4}$$

Assuming the matrix notation, one may write relation (2) as $\Phi_i = \mathbf{C}_i \mathbf{u}_i$, where $\mathbf{C}_i = \left[c_{j(i)},\ \alpha_{j(i)}\right]$ are unknown coefficients and $\mathbf{u}_i = \left\{u_{j(i)},\ f_{j(i)}\right\}$ – degrees of freedom, and the truncated Taylor series (4) with respect to the central node P_i as follows $\mathbf{u}_i \approx \mathbf{\Psi}_i D\mathbf{u}_i$, where $D\mathbf{u}_i = \left\{u_i,\ u_i',\ \ldots,\ u_i^{(p_{\max})}\right\}$ is the vector of all local type derivatives up to $p_{\max}$ order, while the matrix $\mathbf{\Psi}$ is the Taylor series coefficients matrix.

Unknown coefficients $\mathbf{C}$ of the multipoint finite difference formula discretizing the problem (1) may be found from the requirement $\overline{\Phi}_i \equiv \mathbf{C}_i \mathbf{\Psi}_i D\mathbf{u}_i = \mathbf{0}$ and $\mathbf{C}_i\ \mathbf{\Psi}_i = \mathbf{0}$, taking into account their obvious linear dependence. Here $\overline{\Phi}_i$ is a part of Φ_i involving only the terms up to the order $p_{\max}$, resulting from the truncated Taylor series.

Therefore, the coefficients $c_j = c_{j(i)}$ and $\alpha_j = a_{j(i)}$ are computed by solving the following system of equations:

$$\begin{cases} \sum_j c_j \frac{h_j^p}{p!} = 0, & \text{for derivative orders} \quad p = 0,\ 1, \ldots, r-1 \\ \sum_j \left(c_j \frac{h_j^p}{p!} - \alpha_j \sum_q l_q \frac{h_j^{(p-q)_+}}{(p-q)_+!} \right) = 0, & \text{for} \quad p = r, \ldots, p_{\max}, \end{cases} \tag{5}$$

where r is the lowest derivative order in f, $p_{\max}$ is the highest derivative order with imposed zero coefficient value.

In this way e.g. for the 1D b.v. problem

$$a \cdot u' + u'' = f, \quad u_0 = u_N = 0 \tag{6}$$

and tri-nodal stencil, the FD multipoint operator

$$Au_0 + Bu_1 + Cu_2 = \alpha f_0 + \beta f_1 + \gamma f_2$$