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Dispersion analysis of the meshless local boundary integral equation and radial basis integral equation methods for the Helmholtz equation

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ABSTRACT

Numerical solutions of the Helmholtz equation suffer from pollution effect especially for higher wavenumbers. The major cause for this is the dispersion error which is defined as the relative phase difference between the numerical solution of the wave and the exact wave. The dispersion error for the meshless methods can be a priori determined at an interior source node assuming that the potential field obeys a harmonic evolution with the numerical wavenumber.

In this paper, the dispersion errors, in the solution of 2D Helmholtz equation, for two different meshless methods are investigated, the local boundary integral equation method and the radial basis integral equation method. Radial basis functions, with second order polynomials and frequency-dependent polynomial basis vectors are used for the interpolation of the potential field in both methods. The results have been found to be of comparable accuracy with other meshless approaches reported in the literature.

1. Introduction

An accurate solution of the Helmholtz equation is of importance for many modern applications in acoustics such as ultrasonics, microfluidics, aeroacoustics, etc. Simulations involving high frequency waves and large scale industrial setups become cumbersome due to high computational storage and CPU time requirements.

When solving the elliptic equations, it is sufficient to increase the number of elements in the domain linearly in order to achieve certain accuracy; however, this is not the case for acoustic problems due to the unstable nature of the Helmholtz operator. Namely, a linear refinement of the mesh for linear increase in the wavenumber leads to an inaccuracy which grows more than linearly and this is known as the pollution effect [1]. The major component of the pollution is the dispersion error, which is defined as the relative phase difference between the exact and the numerical wave [2]. The dispersion and the pollution of the finite element method (FEM) solutions have been extensively studied. It has been shown that for 1D problems the pollution effect can be completely removed [3] and that it is unavoidable for higher dimensions [1]. However, intensive research has been devoted to reduce the dispersion error in higher dimensions, and thus the pollution [1,4,5]. In the quasi-stabilized FEM [6], the interior stencils of the system matrix are defined such that the number of elements required per wave increases as slowly as possible with increasing wavenumber and in the Galerkin least-squares method [7], a modification to the variational problem can be applied in order to minimize the dispersion.

The dispersion errors for meshless methods in acoustic computations have been reported for the element-free Galerkin method (EFGM) [8–10] and the radial point interpolation method (RPIM) [11]. These studies demonstrated that the dispersion error of the meshless methods is significantly reduced in comparison to the classical FEM. An extensive review has been carried out recently for the minimization of the dispersion with respect to the internal parameters of the aforementioned meshless methods [12].

The meshless local boundary integral equation (LBIE) method was first introduced for the solution of potential problems by Zhu et al. [13] and developed over the years for many engineering problems such as elastostatics, thermoelasticity [14], magnetohydrodynamics [15] and heat conduction [16]. Chen et al. [17] have implemented the LBIE in order to solve acoustic problems. They have reported that the use of frequency-dependent radial basis functions in the formulation leads to lower number of source nodes required per wavelength.

In this work, the LBIE has been implemented using the radial basis functions (RBFs) for the interpolation of the potential field. This method will be referred to as the LBIE–RBF hereafter. Different

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internal parameters of the algorithm such as the radial basis function, the radii of the local subdomains and the polynomials used in the approximations are analysed and the optimization of these parameters is studied in order to reduce the dispersion error. The radius of the local subdomains and the domain of influence in the RBF interpolation are investigated in detail; however, the selection of the RBF is kept the same as in the previous work of Ooi and Popov [18,19].

In this paper, the dispersion errors of the LBIE–RBF and the RBIE are investigated. The two formulations use the RBFs for the approximation of the field variable with an augmentation with polynomial basis vectors. The second order polynomial approximation is used in the computations as well as the well-known frequency-dependent basis vectors [11,12]. In Section 2, the RBF formulation of the LBIE has been outlined. Dispersion error is stated in Section 3 together with the modification to the dispersion formulation for the RBIE. Numerical results to the solutions of 1D and 2D problems showing the dispersive behaviour of the methods and their comparison are introduced in Section 4. Concluding remarks are given in Section 5.

2. Mathematical formulations

In this section, the LBIE–RBF and the RBIE for the solution of the Helmholtz equation are introduced. The derivations of the LBIE–RBF differ from the original LBIE [13] and the differences are discussed. These differences are due to the interpolation procedures of the field variables. In the original implementation of LBIE [13,14], moving least squares approximation (MLSA) is used for the interpolation of the field variables whereas in the LBIE–RBF RBFs are employed.

2.1. The problem

Consider a homogeneous and isotropic domain Ω enclosed by the boundary Γ. The Helmholtz equation governing the potential variable u in Ω is given by the following equation:

$$\nabla^2 u(x)+k^2 u(x)=0, \quad \text{for } x\in\Omega, \tag{1}$$

where $x=(x_1, x_2)$ is the field point coordinate in the two-dimensional Cartesian system and k is the wavenumber. The following boundary conditions are prescribed to the above problem:

$$\begin{aligned} &u(x)=u_o(x), \quad \text{for } x\in\Gamma_1,\\ &\frac{\partial u(x)}{\partial n}=q_o(x), \quad \text{for } x\in\Gamma_2, \end{aligned} \tag{2}$$

where u_o and q_o are the suitably prescribed functions and Γ_1 and Γ_2 are the two non-intersecting parts of Γ such that $\Gamma_1\cup\Gamma_2=\Gamma$.

2.2. The local boundary integral equation method

To solve the problems defined by (1) and (2) using the LBIE–RBF, a set of N_t nodes denoted by ξ_m (for $m=1, 2, \ldots, N_t-1, N_t$) is distributed across $\Omega\cup\Gamma$. A circular subdomain defined by Ω_s and enclosed by the boundary Γ_s is generated. These subdomains, which are centred at each node, may be of different radii and may overlap. For the nodes at the global boundary, the subdomains are defined by the intersection between the subdomain and part of the global boundary. This is shown in Fig. 1.

In each subdomain, the integral representation of (1) is derived. Using the Green's second identity, one obtains

$$c(\xi)u(\xi)=\int_{\Gamma_s}\left[u^*(x;\xi)\frac{\partial u(x)}{\partial n}-u(x)\frac{\partial u^*(x;\xi)}{\partial n}\right]\mathrm{d}\Gamma, \tag{3}$$

where $\xi=(\xi_1,\xi_2)$ is the coordinate of the source point and c is a geometrical coefficient such that $c=0.5$ when ξ is at a smooth part

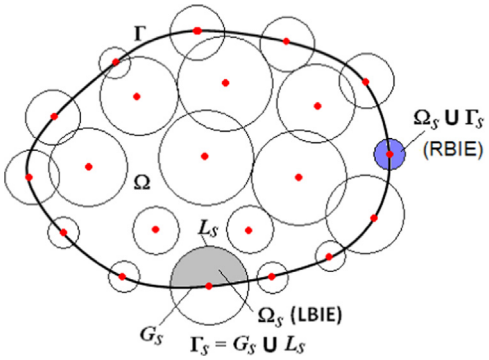

Fig. 1. Source nodes and local sub-domains placed in the solution domain, Ω, and on the global boundary, Γ, for both the LBIE and the RBIE.

of the boundary and $c=1$ when ξ is at the interior. In (3), u^* is the fundamental solution of the Helmholtz equation given by the following equation:

$$u^*(r;\xi)=-\frac{i}{4}H_0^{(2)}(kr), \tag{4}$$

where i is the unit imaginary number, r is the Euclidean distance between the field point x and the source point ξ and $H_0^{(2)}$ is the 2nd order Hankel function of the second kind given by the following equation:

$$H_0^{(2)}(kr)=J_0(kr)-iY_0(kr), \tag{5}$$

where J_0 and Y_0 are the Bessel functions of the first and second kind, respectively.

One of the main features of the LBIE is the removal of the gradient of the unknown function of u along the local boundary Γ_s from the integral equation in (3) [13]. This is accomplished by the use of a companion solution, i.e., an analytical solution u' that satisfies the following Dirichlet problem:

$$\begin{aligned} &\nabla^2 u'(x;\xi)+ku'(x;\xi)=0, \quad \text{in } \Omega_s,\\ &u'(x;\xi)=u^*(x;\xi) \quad \text{on } \Gamma_s. \end{aligned} \tag{6}$$

According to Sladek et al. [16], the problem defined in (6) cannot be solved analytically. As a result, previous attempts at solving the Helmholtz equation using the LBIE [17] were carried out based on the integral equation derived using the fundamental solution of the Laplace equation. This resulted in a domain integral in the integral representation but the Dirichlet problem similar to the one in (6) for the Laplace fundamental solution can be solved analytically.

In the present study, a different approach of solving the Helmholtz equation using the LBIE is suggested. Instead of using the fundamental solution of the Laplace equation, we propose to use a different companion solution, one that satisfies

$$\begin{aligned} &\nabla^2 u'(x;\xi)=0, \quad \text{in } \Omega_s,\\ &u'(x;\xi)=u^*(x;\xi) \quad \text{on } \Gamma_s. \end{aligned} \tag{7}$$

Using $u^{**}=u^*-u'$ as the modified test function and the definition of the companion solution in (8), the integral representation of (1) becomes

$$u(\xi)=-\int_{\Gamma_s}u(x)\frac{\partial u^{**}(x;\xi)}{\partial n}\mathrm{d}\Gamma-\iint_{\Omega_s}k^2u'(x;\xi)u(\xi)\mathrm{d}\Omega, \tag{8}$$

for ξ at the interior. Eq. (8) is expressed purely in terms of the potential field and is free of the gradient.

The integral equation in (8) is valid only for a subdomain that is completely inside the solution domain Ω. For subdomains at the global boundary, the concept of the companion solution does not apply for the part of the local boundary that is formed by the global boundary. In this case, the integral equation is given by the