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An implicit RBF meshless approach for solving the time fractional nonlinear sine-Gordon and Klein–Gordon equations

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ABSTRACT

In this paper, we propose a numerical method for the solution of time fractional nonlinear sine-Gordon equation that appears extensively in classical lattice dynamics in the continuum media limit and Klein–Gordon equation which arises in physics. In this method we first approximate the time fractional derivative of the mentioned equations by a scheme of order $\mathcal{O}(\tau^{3-\alpha})$, $1<\alpha<2$ then we will use the Kansa approach to approximate the spatial derivatives. We solve the two-dimensional version of these equations using the method presented in this paper on different domains such as rectangular and non-rectangular domains. Also, we prove the unconditional stability and convergence of the time discrete scheme. We show that convergence order of the time discrete scheme is $\mathcal{O}(\tau)$. We solve these fractional PDEs on different non-rectangular domains. The aim of this paper is to show that the meshless method based on the radial basis functions and collocation approach is also suitable for the treatment of the nonlinear time fractional PDEs. The results of numerical experiments are compared with analytical solutions to confirm the accuracy and efficiency of the presented scheme.

1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [2,3,21,24,31,53,54,58,66,69,70]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Some of the most applications are given in the book of Oldham and Spanier [68], the book of Podlubny [70] and the papers of Metzler and Klafter [56], Bagley and Trovik [4]. Many considerable works on the theoretical analysis [20,89] have been carried on, but analytic solutions of most fractional differential equations cannot be obtained explicitly. So many authors have resorted to numerical solution strategies based on convergence and stability analysis [7,8,12,23,44,52,57,64,65,62,73,76,80,93,94]. There are several definitions of a fractional derivative of order $\alpha>0$ [68,69]. The two most commonly used are the Riemann–Liouville and Caputo. The difference between the two definitions is in the order of evaluation [67]. We start with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. Now, we give some basic definitions and properties of the fractional calculus theory.

Definition 1. For $\mu\in\mathbb{R}$ and $x>0$, a real function $f(x)$, is said to be in the space C_μ if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C(0,\infty)$, and for $m\in\mathbb{N}$ it is said to be in the space C_μ^m if $f^m\in C_\mu$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha>0$ for a function $f(x)\in C_\mu, \mu\ge -1$, is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt, \quad \alpha>0, \quad x>0, \quad J^0 f(x)=f(x).$$

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Also we have the following properties:

- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
- $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 3. If m be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as

$$ {}_0^C D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{\partial^m u(x,s)}{\partial s^m} (t-s)^{m-\alpha-1}\, ds, & m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & m = \alpha. \end{cases} \tag{1.1}$$

In this paper, we consider the time fractional nonlinear equation

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \nabla^2 u(\mathbf{x},t) - F(u(\mathbf{x},t)) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad t \in [0,T], \tag{1.2}$$

with the initial conditions

$$\begin{aligned} &u(\mathbf{x},0) = \omega(\mathbf{x}), \\ &\frac{\partial u(\mathbf{x},t)}{\partial t}\Big|_{t=0} = \psi(\mathbf{x}), \end{aligned} \quad \mathbf{x} \in \Omega, \tag{1.3}$$

and boundary condition

$$u(\mathbf{x},t) = h(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{1.4}$$

where $u(\mathbf{x},t)$ represents the solute concentration, $f(\mathbf{x},t)$ is source term, $h(\mathbf{x},t)$ is boundary solute concentration, $\omega(\mathbf{x})$, $\psi(\mathbf{x})$ are initial solute concentration, α is the order of time derivative, $\partial^\alpha u(\mathbf{x},t)/\partial t^\alpha$ is the Caputo fractional derivative and is defined as follows:

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\mathbf{x},s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad \alpha \in (1,2), \tag{1.5}$$

and assume $F(u(\mathbf{x},t))$ satisfies the Lipschitz condition with respect to u

$$\left|F(u) - F(\tilde{u})\right| \le L|u - \tilde{u}|, \quad \forall u, \ \tilde{u}. \tag{1.6}$$

Now we consider two important cases of Eq. (1.2):

- Firstly, in Eq. (1.2), if we put $F(u) = \sin(u)$ then we obtain time fractional nonlinear sine-Gordon equation. The main aim of [45] is to present new developments on a 1D lattice model with fractional power-law interatomic interaction defined by fractional values of the parameter s and nonlinear self-interaction potential for classical and quantum mechanical consideration. Also the authors of [45] showed that the dynamics on the 1D lattice can be equivalently presented by the corresponding fractional nonlinear equation in the long-wave limit and they concentrated on the conditions of such equivalence, type of the equations with fractional derivatives and some related properties. As examples of their developments, fractional sine-Gordon and wave-Hilbert nonlinear equations have been found for classical lattice dynamics. Also fractional nonlinear Schrödinger and nonlinear Hilbert–Schrodinger equations have been obtained for quantum lattice dynamics in the long-wave limit. The study of [83] shows that the list of possible applications of fractional equations can be naturally expanded to include non-chaotic and non-random dynamics as well. Authors of [32] applied the homotopy perturbation method for finding approximate analytical solutions of the fractional nonlinear Klein–Gordon equation as a numerical algorithm. The main aim of [39] is to use a fractional subequation method to construct the exact analytical solutions of the space-time fractional Cahn–Hilliard and the fractional nonlinear Klein–Gordon equations.
- Secondly, in Eq. (1.2), if we set $F(u) = -r_1 u - r_2 u^2 - r_3 u^3$ then we get time fractional nonlinear klein–Gordon equation. As mentioned in [28] similar results about fractional Klein–Gordon-type equations have been recently discussed in [29], where an application to the fractional telegraph-type processes has been investigated. The main aim of [28] is to present some exact results related to the fractional Klein–Gordon equation involving fractional powers of the D'Alembert operator also, the authors of the mentioned paper found an exact analytic solution by using the McBride theory of fractional powers of hyper-Bessel operators. A wavelet method for a class of space-time fractional Klein–Gordon equations with constant coefficients is proposed in [36], by combining the Haar wavelet and operational matrix together and efficaciously dispersing the coefficients. The homotopy analysis method is used to construct an approximate solution for the nonlinear space-time fractional derivatives Klein–Gordon equation in [30]. Kurulay [43] proposed the homotopy analysis method to obtain the solution of nonlinear fractional Klein–Gordon equation [61]. Authors of [85] presented a high order finite difference scheme for solving the two-dimensional nonlinear fractional Klein–Gordon equation subject to Neumann boundary conditions. Authors of [12] applied the homotopy analysis method to solve linear fractional problems such as fractional wave, Burgers, Korteweg-de Vries (KdV), KdV-Burgers, and Klein-Gordon equations with initial conditions.

1.1. A brief review of the meshless method

In recent years radial basis functions (RBFs) have been extensively used in different context and emerged as a potential alternative in the field of numerical solution of PDEs. The use of RBFs in the numerical solution of partial differential equations (PDEs) has gained