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Dual reciprocity versus Bessel function fundamental solution boundary element methods for the plane strain deformation of a thin plate on an elastic foundation

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ABSTRACT

Two different direct boundary element methods are used to solve an interior Dirichlet problem for the Navier equations describing the plane strain deformation of a thin plate on an elastic foundation. One method uses dual reciprocity with the Kelvin fundamental solution. The other method develops a fundamental solution that takes into account the effective load term resulting from the elastic foundation. Both methods, related topics, and tradeoffs are described. Test problems and a numerical convergence study are included.

1. Introduction

We consider two boundary element methods for solving the system of partial differential equations describing an interior Dirichlet problem for the plane strain deformation of a thin plate on an elastic foundation. In Section 2 the boundary element equations for plane strain are briefly reviewed. Section 3 presents the dual reciprocity equations for the plane strain deformation of a thin plate on an elastic foundation, extending the basic method as illustrated in [1] to the case of interest in which the load term involves the unknown. Section 4 contrasts the material in Section 3 with a new boundary integral method in which a fundamental solution involving Bessel functions is developed. This development can be compared with that in [2], in which the scalar Helmholtz problem is considered.

Two numerical examples and a numerical convergence study are presented in Section 5. Comparisons of the two methods introduced in the previous sections are summarized in Section 6, and in Section 7 results are reviewed.

2. Boundary element equations for plane strain

We follow the notation and presentation in [3, Section 6.2.1], which is summarized here for completeness. Cartesian tensor notation (Einstein summation notation) is used, so that a repeated letter subscript in an equation implies summation, and differentiation is denoted by commas.

The system of plane strain equations for a linear, homogeneous, isotropic material, expressed in terms of the displacement components (Navier equations) is

$$Gu_{j,kk}+\frac{G}{1-2\nu}u_{k,kj}=b_j, \tag{1}$$

and the surface traction is given by

$$\frac{2G\nu}{1-2\nu}u_{k,k}n_i+G(u_{i,j}+u_{j,i})n_j=p_i,$$

where $i,j,k=1,2$; ν is Poisson's ratio; G is the shear modulus; u_i are the displacement components; p_i are the surface traction components; and b_i are the load components.

The boundary element equations are obtained using a method of weighted residuals in which the weighting function is chosen to be u^*, the Kelvin fundamental solution for (1). Quantities associated with the fundamental solution are indicated with an asterisk superscript. The fundamental solution satisfies

$$Gu^*_{lj,kk}(r)+\frac{G}{1-2\nu}u^*_{lk,kj}(r)=-\delta_{lj}\delta(r) \tag{2}$$

where $j,k,l=1,2$; δ_{lj} is the Kronecker delta; r is the distance between the load and field points; and $\delta(r)$ is the Dirac delta (generalized) function expressed as a function of r. The fundamental solution and the fundamental surface traction have components, respectively,

$$u^*_k=u^*_{1k}+u^*_{2k}, \tag{3}$$

$$p^*_k=p^*_{1k}+p^*_{2k}, \tag{4}$$

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for $k=1,2$. The double subscripts in (3) and (4) use the first subscript for the direction of the Dirac delta function load and the second subscript for the direction of the resulting displacement or surface traction at a field point. The plane strain fundamental solution and fundamental surface traction components are, respectively,

$$u^*_{ij}=\frac{1}{8\pi(1-\nu)G}\left((3-4\nu)\delta_{ij}\ln\left(\frac{1}{r}\right)+r_{,i}r_{,j}\right), \tag{5}$$

$$p^*_{ij}=\frac{-1}{4\pi(1-\nu)r}\left([(1-2\nu)\delta_{ij}+2r_{,i}r_{,j}].\frac{\partial r}{\partial n}-(1-2\nu)(r_{,i}n_j-r_{,j}n_i)\right), \tag{6}$$

where $i,j=1,2$, and $\mathbf{n}=(n_1,n_2)$ is the unit outward normal vector on the boundary [1, p. 268, Eq. 10c], [3, p. 227, Eq. 6.23], [4, p. 444, Eq. B.82], [5, p. 164, Eq. 6.28].

For this Dirichlet problem there are two basic unknown quantities: the surface traction and the interior displacement. The surface traction can be determined first. The surface traction is then used to obtain the interior displacement. To determine the surface traction, Eq. (1) is multiplied by the fundamental solution, u^*, and integrated over the two-dimensional domain. Green's second identity is used, and the boundary integral involving the Dirac delta function from (2) is simplified. The two displacement components evaluated at a load point, $\mathbf{x}^i=(x^i_1,x^i_2)$ are described by the Somigliana equations,

$$c^i_{lk}u^i_k=\int_\Gamma u^{*i}_{lk}p_k\,d\Gamma-\int_\Gamma p^{*i}_{lk}u_k\,d\Gamma-\int_\Omega u^{*i}_{lk}b_k\,da, \tag{7}$$

where $l,k=1,2$; $d\Gamma$ is the differential element of arc length; and da is the differential element of area. The domain is Ω and its boundary is Γ. If the evaluation (load) point $\mathbf{x}^i$ is a smooth boundary point, then $c^i_{lk}=\frac{1}{2}\,\delta_{lk}$. If $\mathbf{x}^i$ is an interior boundary point, then $c^i_{lk}=\delta_{lk}$.

In matrix-vector form, (7) becomes

$$c^i\mathbf{u}^i=\int_\Gamma u^{*i}\mathbf{p}\,d\Gamma-\int_\Gamma p^{*i}\mathbf{u}\,d\Gamma-\int_\Omega u^{*i}\mathbf{b}\,da,$$

with

$$\mathbf{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix},\quad \mathbf{p}=\begin{pmatrix}p_1\\p_2\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix},$$

and

$$u^{*i}=\begin{pmatrix}u^{*i}_{11} & u^{*i}_{12}\\ u^{*i}_{21} & u^{*i}_{22}\end{pmatrix},\quad p^{*i}=\begin{pmatrix}p^{*i}_{11} & p^{*i}_{12}\\ p^{*i}_{21} & p^{*i}_{22}\end{pmatrix}.$$

The vector-valued functions $\mathbf{u}$ and $\mathbf{p}$ are discretized so that

$$c^i\mathbf{u}^i=\sum_{j=1}^{N}\left(\int_{\Gamma_j}u^{*i}\phi^T\,d\Gamma\right)\mathbf{p}^\mathbf{n}-\sum_{j=1}^{N}\left(\int_{\Gamma_j}p^{*i}\phi^T\,d\Gamma\right)\mathbf{u}^\mathbf{n}-\int_\Omega u^{*i}\mathbf{b}\,da,$$

where the jth boundary element is Γ_j, $j=1,2,\ldots N$, and N is the number of boundary elements. The basis function matrix for discretizing the boundary displacement and surface traction is

$$\phi^T=\begin{pmatrix}\phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \cdots & \phi_N & 0\\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 & \cdots & 0 & \phi_N\end{pmatrix},$$

where ϕ_i, $i=1,\ldots,N$, is the ith basis (interpolation) function. For the cases in which piecewise constant or piecewise linear basis functions are used, the number of basis functions equals the number of boundary elements. Solely for convenience in notation, we restrict attention to these cases. The vectors $\mathbf{u}^\mathbf{n}$ and $\mathbf{p}^\mathbf{n}$ are the nodal boundary displacement and nodal surface traction, respectively (the lower case n superscript indicates nodal quantities),

$$\mathbf{u}^\mathbf{n}=(u^1_1,u^1_2,u^2_1,u^2_2,\ldots u^N_1,u^N_2)^T,$$
$$\mathbf{p}^\mathbf{n}=(p^1_1,p^1_2,p^2_1,p^2_2,\ldots p^N_1,p^N_2)^T,$$

where u^i_j is the jth displacement component at the ith boundary node; $i=1,2,\ldots,N$; $j=1,2$; and similarly for the surface traction. The nodes are selected to be at smooth boundary points.

Collocating at each of the load points (boundary nodes), a linear system of $2N$ equations in $2N$ unknown nodal surface traction components is obtained. The linear system can be expressed as

$$G\mathbf{p}^\mathbf{n}=H\mathbf{u}^\mathbf{n}+\left(\int_\Omega u^{*1}\mathbf{b}\,da+\cdots+\int_\Omega u^{*N}\mathbf{b}\,da\right)^T, \tag{8}$$

in which the free (displacement) term coefficients of 1/2 have been incorporated into the main diagonal of the H matrix. The entries of matrix $H=(h_{ij})$ are

$$h_{ij}=\int_\Gamma\begin{pmatrix}p^{*i}_{11} & p^{*i}_{12}\\ p^{*i}_{21} & p^{*i}_{22}\end{pmatrix}\begin{pmatrix}\phi_j & 0\\ 0 & \phi_j\end{pmatrix}d\Gamma,\quad \text{if } i\neq j,$$

and

$$h_{ii}=\int_\Gamma\begin{pmatrix}p^{*i}_{11} & p^{*i}_{12}\\ p^{*i}_{21} & p^{*i}_{22}\end{pmatrix}\begin{pmatrix}\phi_i & 0\\ 0 & \phi_i\end{pmatrix}d\Gamma+\frac{1}{2}I_2,$$

where I_2 is the 2×2 identity matrix. The entries of matrix $G=(g_{ij})$ are

$$g_{ij}=\int_\Gamma\begin{pmatrix}u^{*i}_{11} & u^{*i}_{12}\\ u^{*i}_{21} & u^{*i}_{22}\end{pmatrix}\begin{pmatrix}\phi_j & 0\\ 0 & \phi_j\end{pmatrix}d\Gamma,$$

where $i,j=1,\ldots N$. The indices i and j refer, respectively, to the ith collocation point and to the jth boundary integration region (that part of the boundary on which $\phi_j\neq 0$). The integrands are locally non-zero, each h_{ij} or g_{ij} integration yielding a nonzero result only from boundary elements containing node j.

Since the right-hand side of (8) is a known quantity, that equation can be solved for the unknown nodal surface tractions. Once the surface traction is known (7), collocated at an interior point $\mathbf{x}^i$, can be solved for the interior displacements.

3. Dual reciprocity with the Kelvin fundamental solution applied to the plane strain deformation of a thin plate on an elastic foundation

We now consider the transformation of the domain integral in (7) into an equivalent boundary integral expression. Continuing as in Section 2, material from [3, Section 6.2.1] is summarized here for completeness, and notation from that source is followed closely. The problem of interest is that of the plane strain deformation of a thin plate on an elastic foundation, described by (1) with $b_j=ku_j$,

$$Gu_{j,ll}+\frac{G}{1-2\nu}u_{l,lj}=ku_j, \tag{9}$$

where $j,l=1,2$, and k is a positive constant that accounts for the elastic foundation on which the plate rests. This system can be solved using the boundary element method in more than one way. The first method to be addressed is the dual reciprocity method.

In the dual reciprocity method, the load term, $\mathbf{b}$, is approximated by a linear combination of load basis functions, f^j,

$$b_k\approx\sum_{j=1}^{N+L}f^j\alpha^j_k, \tag{10}$$

where $k=1,2$; N is the number of boundary nodes; L is the number of interior nodes; and α^j_k is an unknown coefficient. Each load basis function, f^j, is specifically chosen so that it has a known (i.e., easily determined) associated particular solution, $\hat{u}^j$, satisfying

$$G\hat{u}^j_{km,ll}+\frac{G}{1-2\nu}\hat{u}^j_{kl,lm}=\delta_{km}f^j,$$