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On the ill-conditioning of the MFS for irregular boundary data with sufficient regularity

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ABSTRACT

In this note, we revisit the issue of ill-conditioning of the method of fundamental solutions (MFS) which was reported by Chen et al. (Eng Anal Bound Elem 30:2006;405–10). Singular value decomposition (SVD) was original proposed by Ramachandran (Commun Numer Methods Eng 18:2002;789–801) to overcome the ill-conditioning of the MFS. The proposed SVD approach given by Ramachandran was somehow contradicted by the results obtained by Chen et al. which stated that Gaussian elimination is a better solver than SVD for non-noisy boundary conditions. For illustration, we provide counter examples to show that the truncated SVD is essential for irregular boundary data and non-smooth domains.
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1. Introduction

The method of fundamental solutions (MFS) was first proposed by Kupradze and Aleksidze [10] in 1964 for solving certain boundary value problems. The MFS started gaining attention after its numerical implementation was proposed by Mathon and Johnston [12] in 1977. In the 1980s, Fairweather and Karageorghis [5] extensively implemented the MFS numerically for solving various types of elliptic boundary value problems. In the 1990s, Golberg and Chen [9] extended the MFS for solving nonhomogeneous problems through the use of radial basis functions and later further extended it to time-dependent problems [2]. Since then, the MFS has re-emerged and attracted great attention in the science and engineering community [6,7]. Three review papers [3,5,8], a series of conferences/workshops, and journal special issues have been specifically dedicated to the development of the MFS.

The major attractions of the MFS are its simplicity and effectiveness for numerically solving various types of partial differential equations. On the negative side, an issue is the poor conditioning of the resultant matrix of the MFS. For an analytic boundary shape and analytic data, this ill-conditioning has little effect on the accuracy of the method. However, in some cases the stability of the MFS has become an issue. To alleviate the ill-conditioning problem, Ramachandran [14] proposed the use of singular value decomposition (SVD). Chen et al. in [1] re-examined some results in [14] and contradicted the conclusion that the SVD is more stable than Gaussian elimination. However, we must stress that the authors in [1] only consider the most ideal cases of sufficiently smooth boundary conditions and boundary geometries. Hence, the assumptions made in [1] were not realistic. Schaback [15] has specifically pointed out that one will always obtain a near perfect solution regardless of the boundary shape when the boundary conditions are generated by an exact harmonic solution. He has further indicated that when the boundary conditions are not harmonic, the rate of convergence of the approximation of the boundary values by harmonic polynomials can be very poor. To avoid confusion, through the rest of this paper we would like to specify that "non-harmonic boundary data" will mean that the solution does not have a harmonic extension to the whole plane. The remark "the MFS makes sense only if the boundary data come from a non-harmonic function or if there is no harmonic extension of the solution without singularities close to the boundary" given by Schaback [15] seems not to be known by most researchers and practitioners. In [4], the authors have verified the above statement by testing the harmonic and non-harmonic boundary data on the circle and the square. Based on the results presented in [4,15], it is expected that the MFS would perform flawlessly in [1] irrespective of the boundary data, shape and the matrix solver. Inspired by the results shown in [4,15], it is the purpose of this note to re-examine the use of the matrix solvers for irregular boundary data and non-smooth boundary shape in the context of the MFS for solving the Laplace equation. Some concluding remarks are placed in the last section.

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2. The method of fundamental solutions

In this note, we will re-examine the issue of using different solvers mentioned in [1,14]. Since the results in this paper can be easily extended to other differential operators and boundary conditions, for simplicity we will focus on the Laplace equation with Dirichlet boundary conditions

$$\Delta u(x,y)=0, \quad (x,y)\in \Omega, \tag{1}$$

$$u(x,y)=g(x,y), \quad (x,y)\in \partial\Omega, \tag{2}$$

where Δ is the Laplacian, $\partial\Omega$ the boundary of the domain Ω, and g a given function.

For more details regarding the numerical implementation of the MFS, we refer readers to [1,14]. Here we only give a brief review of the MFS. The approximate solution $\hat{u}$ of (1)–(2) is given by

$$\hat{u}(x,y)=\sum_{i=1}^{N} c_i \log\|(x,y)-(s_i,t_i)\|, \quad (x,y)\in\overline{\Omega}, \tag{3}$$

where $\{(s_i,t_i)\}_1^N$ are source points on the fictitious boundary located outside the domain $\overline{\Omega}$ and $\|\cdot\|$ is the Euclidean norm. Let $\{(x_i,y_i)\}_1^N$ be N collocation points on the boundary. Then, the coefficients $\{c_i\}_1^N$ are chosen so that (3) satisfies the boundary condition in (2); i.e.,

$$\sum_{i=1}^{N} c_i \log\|(x_j,y_j)-(s_i,t_i)\| = g(x_j,y_j), \quad j=1,2,\ldots,N. \tag{4}$$

The above equations form a system of $N\times N$ equations. In the MFS literature, it is known that the interpolation matrix is dense and ill-conditioned. How to solve (4) accurately and stably has been the subject of intensive research. Following the comments of Schaback [15], we will provide further numerical evidence in the next section to clarify the issue of ill-conditioning of the MFS using different matrix solvers as shown in [1,14].

3. Computational tests

In this section, we present numerical examples of the Laplace equation which includes non-harmonic Dirichlet boundary conditions on smooth and non-smooth boundaries. As mentioned in [4], poor results are achieved only when the boundary data are generated from a non-harmonic function so that the solution does not have a harmonic extension in $\mathbb{R}^2$.

Due to the maximum principle [13], the true solution u and the approximate solution $\hat{u}$ satisfy the error bound

$$\|u-\hat{u}\|_{\infty,\overline{\Omega}} \le \|u-\hat{u}\|_{\infty,\partial\Omega}. \tag{5}$$

This means that the maximum error occurs on the boundary. Hence, we only need to choose the test points on the boundary for the evaluation of the absolute maximum error. The source points are uniformly distributed on a fictitious circle with center at the origin and radius r. The collocation points are also uniformly located on the physical boundary.

In the numerical implementation, for simplicity we only consider Dirichlet boundary condition. We consider the following Dirichlet boundary condition which is a non-harmonic polynomial:

$$g(x,y)=x^2y^3. \tag{6}$$

All the numerical computations in this section are performed using MATLAB. In the legend of all figures in this section, we denote GE as the Gaussian elimination and TSVD as truncated singular value decomposition. For SVD, the given interpolation matrix $\mathbf{A}$ can be decomposed as

$$\mathbf{A}=\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\mathbf{\Sigma}$ is a diagonal matrix with diagonal elements

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge 0.$$

The tolerance of the truncation of TSVD is taken as $tol = \max(size(\mathbf{A}))*\max(\mathbf{\Sigma})*\varepsilon$ where ε is the machine epsilon. This is based on the Moore–Penrose pseudoinverse. The MATLAB function pinv is equivalent to TSVD with truncation tolerance mentioned above.

To illustrate the impact of the boundary shape, we consider two smooth and two non-smooth boundaries. For the smooth boundary, we consider symmetric Cassini and un-symmetric amoeba-like boundary shapes. The parametric equations of these

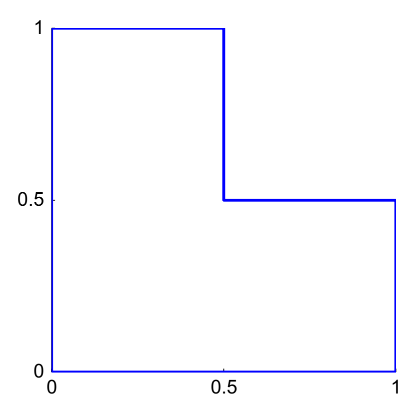

Fig. 2. Profile of L-shape boundary.

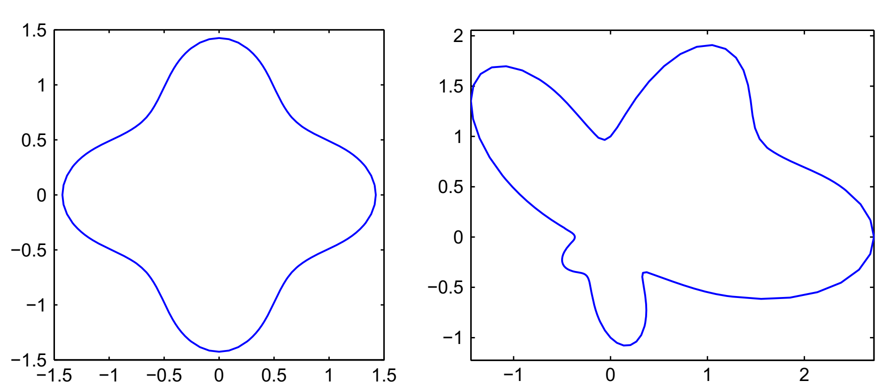

Fig. 1. Profiles of Cassini (left) and amoeba-like (right) boundaries.

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