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Use of Fourier shape functions in the scaled boundary method

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ABSTRACT

The scaled boundary finite element method (SBFEM) is a semi-analytical method, whose versatility, accuracy and efficiency are not only equal to, but potentially better than the finite element method and the boundary element method for certain problems. This paper investigates the possibility of using Fourier shape functions in the SBFEM to form the approximation in the circumferential direction. The shape functions effectively form a Fourier series expansion in the circumferential direction, and are augmented by additional linear shape functions. The proposed method is evaluated by solving three elastostatic and steady-state heat transfer problems. The accuracy and convergence of the proposed method is demonstrated, and the performance is found to be better than using polynomial elements or using an element-free Galerkin approximation for the circumferential approximation in the scaled boundary method.

1. Introduction

The scaled boundary method (SBM) is a semi-analytical method developed relatively recently by Wolf and Song [1]. The method introduces a normalised radial coordinate system based on a scaling centre and a defining curve (usually taken as the boundary). The governing deferential equations are weakened in the circumferential direction and then solved analytically in the normalised radial direction. The SBM has combined the advantages of the Finite Element Method (FEM) and the Boundary Element Method (BEM), and no fundamental solution is required like BEM. In addition, the SBM has been shown to be more efficient than the FEM for problems involving unbounded domains and for problems involving stress singularities or discontinuities [2]. Effective applications of this method have been demonstrated in various problem domains, including fracture problems [3–6] and foundation problems [7–10].

In the scaled boundary method, the discretisation approach used in the circumferential direction has significant influence on the accuracy of the resulting solutions [11]. The most commonly used method for performing this circumferential discretisation is the finite element approach, leading to the method called the scaled boundary finite element method (SBFEM). Vu and Deeks [12–14] investigated the use of higher-order polynomial shape functions in the SBFEM, and demonstrated the SBFEM converged significantly faster under p-refinement than under h-refinement. The development of meshless methods provided another approach to building circumferential approximations for the scaled boundary method. Deeks and Augarde [11] developed a Meshless Local Petrov–Galerkin method scaled boundary method (MLPG-SBM) and He et al. [15] developed an Element-free Galerkin scaled boundary method (EFG-SBM). This work showed that these two meshless scaled boundary methods gave a higher level of accuracy and rate of convergence than the conventional SBFEM using linear or quadratic elements, with the EFG-SBM performing slightly better than the MLPG-SBM.

In this paper, the possibility of using shape functions based on the terms of a Fourier series for the circumferential approximation of the SBFEM is investigated. Fourier interpolations containing trigonometric functions have been applied to both the finite element method (FEM) and the boundary element method (BEM). For example, Guan et al. [16] developed a Fourier series based FEM into for the analysis of tube hydroforming, and showed that this Fourier shape function reduced the number of degrees of freedom required. Khaji et al. [17,18] applied Fourier radial basis functions into the BEM, and showed that of the resulting BEM is much more accurate than the BEM using classic Lagrange shape functions. Although the advantages of Fourier based FEM and BEM have been illustrated in previous work, to date there has been no work reported on the use of Fourier shape functions in the SBFEM.

A new Fourier-based scaled boundary method (F-SBM) is presented in this paper. A set of shape functions based on Fourier series expansion is derived, and augmented with linear shape functions. The new shape functions provide good approximation to both trigonometric and polynomial functions in the circumferential direction of the

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scaled boundary system. In three numerical examples, the F-SBM is used to solve two-dimensional elastostatic and steady-state heat transfer problems. The accuracy and convergence of F-SBM is compared with the conventional SBFEM using both linear and quadratic elements and with the EFG-SBM. Superior performance in terms of both accuracy and convergence is demonstrated.

This paper is organised as follows: The basic equations of scaled boundary method are given in Section 2. Section 3 introduces the Fourier shape functions for use in the scaled boundary method. Three example problems are presented in Section 4 to verify the effectiveness of proposed method, and the paper draws conclusions at the end.

2. The scaled boundary method

The scaled boundary method introduces a normalised radial coordinate system by scaling a defining curve (usually the domain boundary or a part of the domain boundary) relative to a scaling centre (x_0, y_0) selected within the domain or at the intersection of two straight sections of the boundary (Fig. 1). The normalised radial coordinate ξ runs from the scaling centre towards the defining curve, and has values of zero at the scaling centre and unity at the defining curve. The other circumferential coordinate s specifies a distance around the defining curve from an origin on the curve. The scaled boundary and Cartesian coordinate systems are related by the scaling equations

$$x = x_0 + \xi x_s(s) \tag{1}$$

$$y = y_0 + \xi y_s(s) \tag{2}$$

Displacement and stress components are retained in the original Cartesian coordinate directions, while position is specified in terms of the scaled boundary coordinates. An approximate solution is sought in the form

$$\{u_h(\xi,s)\} = \sum_{i=1}^{n} [N_i(s)]u_{hi}(\xi) = [N(s)]\{u_h(\xi)\} \tag{3}$$

This represents a discretisation of the part of the boundary located at $\xi = 1$ with the shape function $[N(s)]$. The unknown vector $\{u_h(\xi)\}$ is a set of n functions analytical in ξ. The shape functions apply for all lines with a constant ξ. (If the scaling centre lies on the boundary, as in Fig. 1, the straight portions of the boundary adjacent to the scaling centre and representing radial lines are not discretised, and in the solution process an analytical solution is found along these lines.)

Mapping the linear operator to the scaled boundary coordinate system using standard methods

$$[L] = [L^1]\frac{\partial}{\partial x} + [L^2]\frac{\partial}{\partial y} = [b^1(s)]\frac{\partial}{\partial \xi} + \frac{1}{\xi}[b^2(s)]\frac{\partial}{\partial s} \tag{4}$$

where $[b^1(s)]$ and $[b^2(s)]$ are dependent only on the boundary definition.

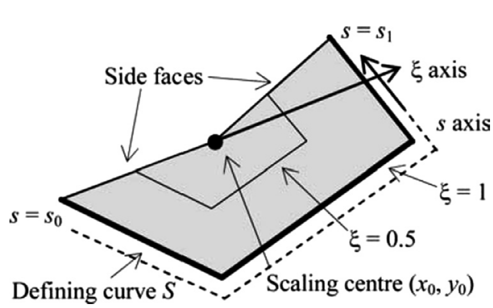

Fig. 1. Bounded domain with side faces showing scaled boundary coordinate system.

The stresses are obtained by multiplying the strains (obtained form the displacement field using the linear operator) by the elasticity matrix $[D]$ in the form

$$\{\sigma(\xi,s)\} = [D]\{\varepsilon(\xi,s)\} = [D][B^1(s)]\{u(\xi)\}_{,\xi} + \frac{1}{\xi}[D][B^2(s)]\{u_h(\xi)\} \tag{5}$$

where

$$[B^1(s)] = [b^1(s)][N(s)] \tag{6}$$

$$[B^2(s)] = [b^2(s)][N(s)]_{,s} \tag{7}$$

In this case the virtual work statement becomes

$$\int_V \{\delta\varepsilon(\xi,s)\}^T\{\sigma_h(\xi,s)\}dV - \int_S \{\delta u(s)\}^T\{t(s)\}ds = 0 \tag{8}$$

where the first term represents the internal work and the second term the external work, and $\{t(s)\}$ is the external force vector.

The virtual strain field is of the form (analogous to Eq. (5))

$$\{\delta\varepsilon(\xi,s)\} = [B^1(s)]\{\delta u(\xi)\}_{,\xi} + \frac{1}{\xi}[B^2(s)]\{\delta u(\xi)\} \tag{9}$$

where $\{\delta u(\xi)\}$ is virtual displacement and

$$dV = |J|\xi d\xi ds \tag{10}$$

where $|J|$ is the Jacobian at the boundary $(\xi = 1)$.

Substituting Eqs. (5), (9) and (10), integrating the area integrals containing $\{\delta u(\xi)\}_{,\xi}$ with respect to ξ using Green's Theorem, and introducing the coefficient matrices

$$[E^0] = \int_S [B^1(s)]^T[D][B^1(s)]|J|ds \tag{11}$$

$$[E^1] = \int_S [B^2(s)]^T[D][B^1(s)]|J|ds \tag{12}$$

$$[E^2] = \int_S [B^2(s)]^T[D][B^2(s)]|J|ds \tag{13}$$

the virtual work equation may be expressed as

$$\int_V \{\delta\varepsilon(\xi,s)\}^T\{\sigma_h(\xi,s)\}\mathrm{d}V = \{\delta u\}^T\left\{[E^0]\{u_h\}_{,\xi} + [E^1]\{u_h\}\right\} - \int_0^1 \{\delta u(\xi)\}^T\left\{[E^0]\xi\{u_h(\xi)\}_{,\xi\xi} + [[E^0]+[E^1]^T - [E^1]]\{u_h(\xi)\}_{,\xi} - [E^2]\frac{1}{\xi}\{u_h(\xi)\}\right\}\mathrm{d}\xi \tag{14}$$

On substitution of Eq. (3), the external virtual work term becomes

$$\int_S \{\delta u(s)\}^T\{t(s)\}ds = \{\delta u\}^T\int_S \{N(s)\}\{t(s)\}ds = \{\delta u\}^T\{P\} \tag{15}$$

Thus the complete virtual work equation becomes

$$\{\delta u\}^T\{[E^0]\{u_h\}_{,\xi} + [E^1]\{u_h\}\} - \{\delta u\}^T\{P\} - \int_0^1 \{\delta u(\xi)\}^T\left\{[E^0]\xi\{u_h(\xi)\}_{,\xi\xi} + [[E^0]+[E^1]^T - [E^1]]\{u_h(\xi)\}_{,\xi} - [E^2]\frac{1}{\xi}\{u_h(\xi)\}\right\}d\xi = 0\} \tag{16}$$

In order for Eq. (16) to be satisfied for all $\{\delta u(\xi)\}$ (implying that equilibrium it closely satisfied in the radial direction and in the approximate sense in the circumferential direction), both of the following conditions must be satisfied.

$$\{P\} = [E^0]\{u_h\}_{,\xi} + [E^1]^T\{u_h\} \tag{17}$$

$$[E^0]\xi^2\{u_h(\xi)\}_{,\xi\xi} + [[E^0]+[E^1]^T - [E^1]]\xi\{u_h(\xi)\}_{,\xi} - [E^2]\{u_h(\xi)\} = \{0\} \tag{18}$$

By inspection, solutions to the homogeneous set of Euler–Cauchy differential equations represented by Eq. (18) must be of