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Efficient computation of the Green's function and its derivatives for three-dimensional anisotropic elasticity in BEM analysis

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ABSTRACT

An alternative scheme to compute the Green's function and its derivatives for three dimensional generally anisotropic elastic solids is presented in this paper. These items are essential in the formulation of the boundary element method (BEM); their evaluation has remained a subject of interest because of the mathematical complexity. The Green's function considered here is the one introduced by Ting and Lee [Q. J. Mech. Appl. Math. 1997; 50: 407–26] which is of real-variable, explicit form expressed in terms of Stroh's eigenvalues. It has received attention in BEM only quite recently. By taking advantage of the periodic nature of the spherical angles when it is expressed in the spherical coordinate system, it is proposed that this Green's function be represented by a double Fourier series. The Fourier coefficients are determined numerically only once for a given anisotropic material; this is independent of the number of field points in the BEM analysis. Derivatives of the Green's function can be performed by direct spatial differentiation of the Fourier series. The resulting formulations are more concise and simpler than those derived analytically in closed form in previous studies. Numerical examples are presented to demonstrate the veracity and superior efficiency of the scheme, particularly when the number of field points is very large, as is typically the case when analyzing practical three dimensional engineering problems.

1. Introduction

For elastostatics, the fundamental solution, otherwise also often referred to as the Green's function, is the solution of the displacements at a field point due to a unit point load at a source point in an infinite body. It is denoted by $\mathbf{U}(\mathbf{x})$ here, where $\boldsymbol{x}=(x_1, x_2, x_3)$ in the Cartesian coordinate system. The availability of $\mathbf{U}(\mathbf{x})$ and its derivatives is a key requirement in the direct formulation of the boundary element method (BEM), the method of fundamental solutions and some other meshless methods. In the conventional displacement-BEM formulation, the Green's function and its first-order derivatives are used in the derivation of the boundary integral equation (BIE) for the displacements and tractions on the surface of the domain. Higher order derivatives are required for evaluating the stresses at interior points of the body via Somigliana's identity. These higher order derivatives are also required in traction-BIE and hypersingular BEM formulations. For practical three-dimensional problems in elastic stress analysis, the number of evaluations of $\mathbf{U}(\mathbf{x})$ and its derivatives is typically in the order of 10^6 and higher. Thus, developing an efficient scheme for the computation of these quantities is very important for the BEM to be a successful numerical tool for solving practical engineering problems.

For isotropic elastostatics, the Green's function and its derivatives can be expressed in relatively simple explicit forms; the computational effort to evaluate them has therefore never been a serious issue. Although this is also the case for 2D anisotropic elasticity, the same cannot be said for 3D general anisotropic solids due to the mathematical complexity. Indeed, the numerical evaluation of $\mathbf{U}(\mathbf{x})$ and its derivatives for 3D general anisotropic elasticity has remained a subject of great interest in the BEM community over the past few decades. The fundamental solution $\mathbf{U}(\mathbf{x})$ for a 3D generally anisotropic solid was first presented in 1947 by Lifschitz and Rozentsweig [1]. It is expressed as a line integral around a unit circle, the integrand of which contains the Christoffel matrix defined in terms of the elastic constants. Numerous attempts have since been made to analytically evaluate this integral and its derivatives into as simple and explicit a form as possible, see, e.g., [2–7].

Wilson and Cruse [8] were the first to implement a numerical formulation of the BEM for 3D stress analysis of a generally anisotropic solid using the form of $\mathbf{U}(\mathbf{x})$ presented in [1]. Their algorithm involves numerically evaluating the contour integral and building a large database of the point load solutions and their derivatives for a given material. In the formation of the linear

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equations when solving the BIE in the BEM formulation, bivariate cubic spline interpolation of these pre-calculated values is employed. This approach is computationally quite demanding and its accuracy for treating highly anisotropic materials has also been called into question. Since the pioneering work in [8], several other schemes to evaluate these quantities have been proposed for the development of the BEM to treat 3D generally anisotropic solids. Some are more elegant and efficient than others; they include the perturbation expansion technique, the dual reciprocity approach, the method of residue calculus, the Radon transform approach and, more recently, the explicit real variable algebraic solution scheme. References [9–17] represent a selection of BEM works using these different approaches; more extensive reviews are given in [16,17] by the authors, as well as others in the list. It is perhaps also worth noting that other fast Green's function algorithms, although not specific to BEM implementation nor 3D general anisotropic elasticity, have also been proposed in the literature. They include the method of Fourier transforms [18], the divergent series approach [19], the B-spline and other interpolation schemes of pre-calculated solutions [20,21].

Among the various forms of $\mathbf{U}(\mathbf{x})$ that have been introduced is one that is real variable, fully explicit and algebraic in form. It was derived using Stroh's formalism by Ting and Lee [6] and is expressed in terms of Stroh's eigenvalues. Lee [7] subsequently showed how the derivatives of $\mathbf{U}(\mathbf{x})$ may be generated, presenting them in the form of higher order tensors. These solutions, however, did not receive much attention in the BEM community until very recently. Because of their explicit forms, they could be implemented into, for example, existing BEM codes for 3D isotropic elastostatics in a relatively direct manner. This was first carried out by Tavara et al. [15] for the special case of transverse isotropy, and by the present authors [16,17] for the case of fully general anisotropy. They were also employed by Buroni and Saez [22] to treat anisotropic materials exhibiting magneto–electro–elastic coupling. The BEM implementation reported in [16,17], although fairly straightforward, revealed the relative inefficiency of computing the higher order derivatives of $\mathbf{U}(\mathbf{x})$ due to the presence of the very high order tensors. For example, 10th order tensors are present in the 2nd order derivatives of $\mathbf{U}(\mathbf{x})$. This led Lee [23] to revisit the problem from which it is shown that by carrying out the differentiation of $\mathbf{U}(\mathbf{x})$ in the spherical coordinate system as an intermediate step instead of direct differentiation with respect to the Cartesian coordinates, simpler analytical forms of the derivatives can be obtained that obviates the introduction of very high order tensors. Using this revised approach and the residue theorem for high order poles, the present lead authors derived the 1st and 2nd order derivatives of $\mathbf{U}(\mathbf{x})$ and reported their successful implementation into BEM in [24–26]. The better computational performance with these expressions over the previous formulations is also demonstrated in [25,26]. Although they are as relatively straightforward as before, these expressions are rather elaborate in form and their implementation is somewhat tedious and involved. For even higher order derivatives of $\mathbf{U}(\mathbf{x})$ that are required in, for example, hypersingular BEM formulations, it can be expected that this will be even more so.

In this paper, an alternative scheme is proposed to numerically evaluate the explicit form Green's function of Ting and Lee [6], and its derivatives. Instead of computing directly the explicit algebraic expression of $\mathbf{U}(\mathbf{x})$ and its derivatives as derived in [6] and [24], the fundamental solution is first expressed as a double Fourier series. This is possible by virtue of the periodic nature of the spherical angles when Ting and Lee's solution is expressed in the spherical coordinate system. The Fourier coefficients are evaluated numerically, and this is done once only for a given material, independent of the number of field points in the solution domain. The derivatives of $\mathbf{U}(\mathbf{x})$ can be directly obtained by spatial differentiation on the double Fourier series. This makes the scheme very efficient and attractive for practical engineering problems, as the number of field points at which the numerical values of $\mathbf{U}(\mathbf{x})$ and its derivatives are required in the BEM analysis is typically very large indeed. To demonstrate the veracity and accuracy of this scheme, the computed Green's function and its derivatives for sample points of an anisotropic material are compared with the results obtained by the other forms of the same $\mathbf{U}(\mathbf{x})$ and its derivatives previously employed in [17], [24]. The numerical efficiency of using the various approaches is also compared to demonstrate the superior efficiency of the present scheme. To this end, it is useful to briefly review the fundamental solution $\mathbf{U}(\mathbf{x})$ as derived by Ting and Lee [7], and its derivatives, as presented in [17] and [24]. This will help to provide a better perspective of the different schemes as the present Fourier series approach is introduced.

2. Green's function of 3D generally anisotropic solids

As mentioned earlier, the Green's function $U_{ij}(P, Q) \equiv \mathbf{U}(\mathbf{x})$ is defined as the displacement response in the x_i-direction at the field point Q due to a unit load applied in the x_j-direction at P in a homogeneous infinite body. The present approach is to express the analytically exact and explicit form of the Green's function derived by Ting and Lee [6] as a double Fourier series. This fundamental solution will first be reviewed below.

For a generally anisotropic material, the point load solution for the displacement field may be written as [1–3]

$$U_{ij} = \frac{1}{8\pi^2 r}\int_0^{2\pi} \mathbf{Z}^{-1} d\psi, \tag{1}$$

where r is the radial distance between the source point P at the local origin $\mathbf{x}=\mathbf{0}$ and the field point Q at $\boldsymbol{x}=(x_1, x_2, x_3)$. In Eq. (1), the integral is taken around the unit circle $|\mathbf{n}^*| = 1$ on the oblique plane normal to $\boldsymbol{x}$ as shown in Fig. 1. The unit vector $\mathbf{n}^*$ on the oblique plane can be written in terms of an arbitrary parameter ψ as follows

$$\mathbf{n}^* = \mathbf{n}\cos\psi + \mathbf{m}\sin\psi, \tag{2}$$

where the vectors $\mathbf{n}$, $\mathbf{m}$ along with $\mathbf{x}/r$ form a right-handed triad $[\mathbf{n},\mathbf{m},\mathbf{x}/r]$. With reference to Fig. 1, the vectors $\mathbf{n}$ and $\mathbf{m}$ can be

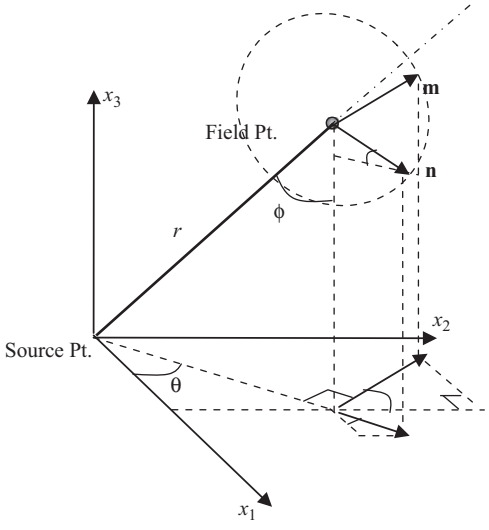

Fig. 1. The unit circle on an oblique plane at the field point; definition of the unit vectors $\mathbf{m}$ and $\mathbf{n}$.