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# Joint Probability Distribution and the Minimum of a Set of Normalized Random Variables

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**Abstract**

Suppose that $n$ types of components $M_1$, $M_2$… $M_n$ are combined to form and integrated object $I$ and suppose that $y$ units of the integrated object are required to be formed. Assuming that not all components can be used in forming the integrated objects, let $q_j$ be the percentage of usable components of the $j^{th}$ type, a random variable having a probability density function $f_j(q_j)$. Let $w_j$ be the normalized random variable obtained from $q_j$ by $w_j = q_j/\mu_j$, where $\mu_j$ is the expected value of $q_j$. Consider the random variable $W=Min\{w_j, 1 \leq j \leq n\}$. This paper describes the joint probability distribution of the set of the normalized random variables and determines the probability distribution of the minimum $W$ of this set. The expected value of $W$ is key to determining the number of components needed to form the $y$ integrated objects. A special case is presented where the percentages of usable components are uniformly distributed. The problem is applied to a production model.


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## 1. Introduction

Suppose that $n$ types of components $M_1, M_2, \ldots, M_n$ are combined to form one integrated object $I$ and suppose that $y$ integrated objects are to be formed. It is assumed that each integrated object requires one component of each type and not all components can be used in forming the integrated objects. That is, each set of components contains a percentage of usable items. This percentage is a random variable having a known probability density function. The number of components of each type needed to form the required $y$ integrated objects is obtained by determining the probability distribution of the minimum of a set of normalized random variables obtained from the percentages of

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usable components. This provides an exact solution to a model introduced by Khan and Jaber (2011) who proposed an approximate solution based on a very restrictive and unrealistic assumption.

The aim of this paper is to solve the problem of determining the number of components of each type needed to form the $y$ integrated objects. A special case is presented where the number of components is two and their corresponding percentages of usable components are uniformly distributed. A numerical example is provided to illustrate the special case. The general solution is applied to production model that can be used to provide an exact solution to the model of Khan and Jaber (2011).

## 2. The Statistical Problem

Consider a set of $n$ independent random variables $\{q_1, q_2, \ldots, q_n\}$, and let $f_j(q_j)$ be the probability density function of $q_j$, $1 \le j \le n$. For each $q_j$, define the normalized random variables to be $w_j = q_j/\mu_j$, where $\mu_j$ is the expected value of $q_j$. Let $g_j(w_j)$ denote the probability density function of the random variable $w_j$ and let $G_j(t)$ be its a cumulative distribution; i.e, $G_j(t) = \text{Prob}(w_j \le t)$. Then, $w_1, w_2, \ldots, w_n$ are also independent and their joint probability distribution $g(w_1, w_2, \ldots, w_n)$ is the product $g_1(w_1)g_2(w_2)\ldots g_n(w_n)$. Also, the expected value of $w_j$ is equal to one and the its standard deviation is $\mu_j/\sigma_j$. Define the random variable $W$ to be the minimum of $w_1, w_2, \ldots, w_n$. That is, $W = Min\{w_j, 1 \le j \le n\} = Min\{q_j/\mu_j, 1 \le j \le n\}$. The cumulative distribution $G(t)$ of $W$ is given by

$$G(t) = \text{Prob}(W \le t) = \int\int\ldots\int_B g(w_1, w_2, \ldots, w_n)dw_1 dw_2 \ldots dw_n,$$

where the region $B$ is the intersection between the domain of $g(w_1, w_2, \ldots, w_n)$ and the region representing $W \le t$. Using the fact that $w_1, w_2, \ldots, w_n$ are independent and the fact that $B$ is a cube-like region, we have

$$\begin{aligned} G(t) = \text{Prob}(W \le t) &= \int\int\ldots\int_B g(w_1, w_2, \ldots, w_n)dw_1 dw_2 \ldots dw_n \\ &= \left(\int_{-\infty}^{t} g(w_1)dw_1\right)\left(\int_{-\infty}^{t} g(w_2)dw_2\right)\ldots\left(\int_{-\infty}^{t} g(w_n)dw_n\right) = G_1(t)G_2(t)\ldots G_n(t). \end{aligned} \tag{1}$$

Once the cumulative distribution $G(t)$ is determined, the probability density function $g(t)$ is obtained by differentiating $G(t)$ and the expected value $\mu$ of $W$ can be calculated by

$$E[W] = \mu = \int_{-\infty}^{\infty} tg(t)dt. \tag{2}$$

To illustrate, we consider the case where each $q_j$ is uniformly distributed over $[a_j, b_j]$, so that $\mu_j = E[q_j] = (a_j+b_j)/2$. Then, $w_j = q_j/\mu_j$ is uniformly distributed over $[a_j/\mu_j, b_j/\mu_j]$ and $E[W_j] = 1$. Since $E[W_j] = 1$ is the midpoint, the interval $[a_j/\mu_j, b_j/\mu_j]$, we can write $[a_j/\mu_j, b_j/\mu_j] = [1-m_j, 1+m_j]$, where $m_j = (b_j-a_j)/(a_j+b_j)$. Hence, the probability density function of $w_j$ is $g_j(w_j) = 1/(2m_j)$ and its cumulative probability density function is given by

$$G_j(t) = \begin{cases} 0 & if \quad t \le m_j \\ \dfrac{t-m_j+1}{2m_j} & if \quad 1-m_j \le t \le 1+m_j \\ 1 & if \quad t \ge 1+m_j \end{cases}. \tag{3}$$

From (1), $G(t)$ is obtained from $G_1(t)G_2(t)\ldots G_n(t)$ and the probability density function of $W$ is $g(t) = dG(t)/dt$. Equation (2) can then be used to calculated the expected value of $W$.