Engineering Analysis with Boundary Elements 36 (2012) 1923–1930

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A new radial basis function for Helmholtz problems

J. Lin [a,b], W. Chen [b,*], K.Y. Sze [a]

[a] Department of Mechanical Engineering, The University of Hong Kong, Pokfulam, Hong Kong SAR, PR China
[b] College of Mechanics and Materials, Hohai University, Nanjing, PR China

ARTICLE INFO

Article history:
Received 30 July 2011
Accepted 25 July 2012
Available online 24 August 2012

Keywords:
Radial basis function
Regularization technique
Helmholtz problem

ABSTRACT

In this paper, a new radial basis function (RBF) is proposed to solve Helmholtz problems in the traditional collocation method. Since the matrix equation arising from the RBF interpolation is ill-conditioned, a regularized singular value decomposition method is used to obtain a more accurate solution. Numerical examples of both direct and inverse problems are presented to demonstrate the effectiveness and applicability of the proposed RBF versus the traditional multiquadric RBF.

1. Introduction

A large variety of RBFs or RBF-based methods have been proposed. Commonly used RBFs, such as multiquadric (MQ) [1,2], inverse multiquadric [3], and thin plate spline [4,5], have been widely studied. The popular RBF-based numerical methods include the method of fundamental solutions (MFS) [6,7], boundary collocation method [8,9], regularized meshless method [10,11], radial basis function networks [12,13], radial basis collocation method [14,15], boundary distributed source method [16], and boundary knot method (BKM) [17,18], etc. In the past several decades, the above RBFs or RBF-based methods have been applied to solve heat transfer problems [19,20], 1D and 2D nonlinear Burgers' equation [21,22], shallow water equation for tide and currents simulation [23], harmonic elastic and viscoelastic problems [24], among others.

The initial development of applying RBFs to solve partial differential equation began from the pioneering work of Kansa [25,26], named as Kansa's method. In this method, RBFs are directly used as the basis to approximate the solutions by enforcing the governing equation and boundary conditions on collocation points. The MQ was first developed by Hardy [27] as a multi-dimensional scattered interpolation in approximating the gravitational field of the earth. It was not recognized by most of the researchers until Franke [28] published a paper in which the accuracy, efficiency, storage, and the ease of implementation of 29 interpolation methods were evaluated and MQ was ranked as the overall best. When applying some RBF-based methods, such as the MFS and BKM, to non-homogeneous problems [29,30], one needs to resort to a two-step method. That is to say, one should first approximate the particular solution by dual reciprocity method or other methodologies, and then derive the general solution for the corresponding homogeneous problems. Compared with these numerical methods, the radial basis collocation method [31–34] is a single-step method for both homogeneous and non-homogeneous problems. However, the accuracy of the method is highly sensitive to the choice of RBF.

In this study, a new RBF to be used in the radial basis collocation method is proposed for the both direct and inverse Helmholtz problems. It is based on the general solution of Helmholtz equation. This type of RBF is constructed by a heuristic approach without rigorous mathematical analysis. To illustrate its effectiveness and efficiency, several direct and inverse problems are considered. In the direct problems, the coefficient matrix generated by the new RBF is often ill-conditioned as those generated by other traditional RBFs [35,36]. In the inverse problems, we only consider the classical Cauchy problems in which boundary conditions for both the solution and its normal derivative are prescribed only on a portion of the boundary, whilst no information is available on the remaining part. So, we should reconstruct the solution on the unaccessible boundary and in the domain. The Cauchy problem is much more difficult to solve both analytically and numerically than the direct problem, since the solution does not satisfy the general conditions of well-posedness. The solution is not a continuous function of the boundary data and a small error in the accessible data may result in an enormous error in the numerical solution, this kind of problem is ill-posed [37]. We cannot use direct approach, such as the Gauss elimination method, in order to solve the system of linear equations which arises from the discretization of the problem. To handle ill-conditioned or ill-posed problems, many regularization techniques have been adopted [37–39]. Here, we extend the new RBF combined with the damped singular value decomposition (DSVD) regularization technique to Cauchy problems. The generalized cross

* Corresponding author.
E-mail address: chenwen@hhu.edu.cn (W. Chen).

0955-7997/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.enganabound.2012.07.010

validation (GCV) [37] is one of the strategies to estimate an appropriate regularization parameter of the DSVD and is employed in our numerical experiments.

The rest of the paper is organized as follows. In Section 2, formulations of the direct and inverse problems are briefly reviewed whereas the new RBF would be introduced. The DSVD under parameter choice of GCV is described in Section 3. In Section 4, four numerical examples are employed to study the accuracy, efficiency, convergence and the numerical conditioning of or related to the new RBF. Section 5 concludes this study with some remarks.

2. Problem description and the new radial basis function

2.1. Direct and inverse problems

We consider the following non-homogeneous Helmholtz equation

$$\nabla^2 u(x,y)+k^2 u(x,y)=f(x,y) \quad \text{in } \Omega, \tag{1}$$

where ∇^2 is the Laplacian, k the wave-number, Ω represents a simply connected domain in R^2, and $f(x,y)$ the source term.

For direct problems under investigation require solving Eq. (1) subjected to the following boundary conditions:

$$u(x,y)=g(x,y) \quad \text{on } \Gamma_1, \tag{2}$$

$$\frac{\partial u(x,y)}{\partial n}=h(x,y) \quad \text{on } \Gamma_2, \tag{3}$$

where $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\partial u/\partial n$ denotes the outward normal derivative of u. Lastly, $g(x,y)$ and $h(x,y)$ are the measured Dirichlet and Neumann data on boundaries Γ_1 and Γ_2, respectively.

For Cauchy problems, the boundary condition is not known on the whole boundary $\partial\Omega$

$$u(x,y)=g(x,y) \quad \text{on } \Gamma_1, \tag{4}$$

where Γ_1 is the accessible part of the boundary $\partial\Omega$. The governing equation (1) subjected to only the boundary condition (4) is mathematically under-determined, and additional data must be supplied to fully determine it. The additional data available is given by a boundary condition different from that given by Eq. (4):

$$\frac{\partial u(x,y)}{\partial n}=h(x,y) \quad \text{on } \Gamma_1. \tag{5}$$

Note that in this case, the accessible part of the boundary Γ_1 is overspecified, since two different types of boundary conditions are prescribed on it. A necessary condition for the above Cauchy problem (1), (4) and (5) to be identifiable is that the known boundary part Γ_1 is larger than the under-specified boundary part $\Gamma_2=\partial\Omega/\Gamma_1$. And in this study, we focus on determining the underprescribed functions on the inaccessible boundary Γ_2 and in the solution domain.

2.2. The new radial basis function

At first, general solution of Eq. (1) is as follows:

$$\varphi(\mathbf{x},\mathbf{x}_j)=J_0(kr), \tag{6}$$

where J_0 denotes the zeroth order Bessel function of the first kind, $r=r(\mathbf{x},\mathbf{x}_j)$ is the Euclidian distance between the general points $\mathbf{x}=(x,y)$ and the origin of the RBF $\mathbf{x}_j=(x_j,y_j)$, k the wave-number. In order to solve the nonhomogeneous problem, we propose the following radial basis function:

$$\varphi(\mathbf{x},\mathbf{x}_j)=J_0(k(r^2+C^2)^{1/2}), \tag{7}$$

where C is an empirically chosen shape parameter. It can be seen that the well-known MQ function $(r^2(\mathbf{x},\mathbf{x}_j)+C^2)^{1/2}$ is an argument of this new RBF. The approximation solution is expressed by linear combination of the new RBF (7),

$$u_N(\mathbf{x})=\sum_{j=1}^{N}\alpha_j\varphi(\mathbf{x},\mathbf{x}_j), \tag{8}$$

where α_j is the unknown coefficient to be determined. In our computations, the collocation points to enforce the governing equation or the boundary conditions are identical to the RBF origins. We use $\{x_j\}_1^{N_I}$, $\{x_j\}_{N_I+1}^{N_I+N_D}$ and $\{x_j\}_{N_I+N_D+1}^{N}$ to denote the collocation points in Ω, on Γ_1 and on Γ_2, respectively. Hence, the following linear algebraic equations on α_j's are resulted:

$$\sum_{j=1}^{N}\alpha_j(\nabla^2\varphi(\mathbf{x}_i,\mathbf{x}_j)+k^2\varphi(\mathbf{x}_i,\mathbf{x}_j))=f(\mathbf{x}_i), \quad i=1,2,\ldots,N_I, \tag{9}$$

$$\sum_{j=1}^{N}\alpha_j\varphi(\mathbf{x}_i,\mathbf{x}_j)=g(\mathbf{x}_i), \quad i=N_I+1,\ldots,N_I+N_D, \tag{10}$$

$$\sum_{j=1}^{N}\alpha_j\frac{\partial\varphi(\mathbf{x}_i,\mathbf{x}_j)}{\partial n}=h(\mathbf{x}_i), \quad i=N_I+N_D+1,\ldots,N, \tag{11}$$

which can be written in the following matrix form:

$$[A_{ij}][\alpha_j]=[b_i]. \tag{12}$$

The coefficients matrix $[A_{ij}]$ is often ill-conditioned for both direct and inverse problems. With an ill-conditioned matrix, the predictions would be unstable especially when the input data contains noise [37]. In this context, regularization methods have been used to remedy the instability and accuracy loss in the solution of ill-conditioned matrix [37,40,41]. In this paper, we shall employ the DSVD under parameter choice of GCV which is introduced in the following section.

3. Regularization method and regularization parameter

Before presenting the regularization method and regularization parameter, we introduce the singular value decomposition (SVD) of the coefficient matrix in (12),

$$A=W\Sigma V^T, \tag{13}$$

where $W=[w_1,w_2,\ldots,w_N]\in R^{N\times N}$, $W^TW=I_N$ and $V=[v_1,v_2,\ldots,v_N]\in R^{N\times N}$, $V^TV=I_N$ and I_N denotes the N-th order identity matrix. The singular values of A are the diagonal entries of $\Sigma=\text{diag}(\sigma_1,\sigma_2,\ldots,\sigma_N)$ which has non-negative diagonal elements appearing in non-increasing order such that

$$\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_N\ge 0. \tag{14}$$

The column vectors w_i and v_i are, respectively, left- and right-singular vectors for the corresponding singular values. Using the SVD, it is easy to get the solution to (12)

$$\alpha=\sum_{i=1}^{N}\frac{w_i^T b v_i}{\sigma_i}. \tag{15}$$

Remark: The conventional L_2 condition number of A is defined as $\text{Cond}(A)=\sigma_1/\sigma_N$, in which σ_1 and σ_N are the largest and smallest singular values of A.

3.1. Regularization method

The Damped Singular Value Decomposition (DSVD) is based on the SVD and the Tikhonov regularization (TR) technique [37]. The idea of TR is to define the regularized solution to (12) by the