Engineering Analysis with Boundary Elements 36 (2012) 1931–1941

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Boundary integral equations and the boundary element method for fracture mechanics analysis in 2D anisotropic thermoelasticity

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A R T I C L E I N F O

Article history:
Received 6 January 2012
Accepted 17 July 2012
Available online 29 August 2012

Keywords:
Anisotropic
Thermoelastic
Crack
Stress intensity factor
Boundary integral equation

A B S T R A C T

This paper develops the Somigliana type boundary integral equations for fracture of anisotropic thermoelastic solids using the Stroh formalism and the theory of analytic functions. In the absence of body forces and internal heat sources, obtained integral equations contain only curvilinear integrals over the solid's boundary and crack faces. Thus, the volume integration is eliminated and also there is no need to evaluate integrals over the contours in the mapped temperature domain as it was done before. In addition to finite solids, the case of an infinite anisotropic medium with a remote thermal load is also studied. The dual boundary element method for fracture of anisotropic thermoelastic solids is developed based on the obtained boundary integral equations. Presented numerical examples show the validity and efficiency of the obtained equations in the analysis of both finite and infinite solids with cracks.

1. Introduction

Modern composite materials are widely used in the engineering design due to their useful properties, which allow reduction of weight, yet keeping the strength and rigidity of the product. Composite materials are, in general, anisotropic, which leads to the development of analytical and numerical approaches for the analysis of strength and fracture of anisotropic structural elements, including the study of stress intensity factors at crack tips. Among the numerical approaches, the boundary element method (BEM) is notable for high accuracy and performance due to its semi-analytical nature and the use of only boundary mesh. Sollero and Aliabadi [1], Pan and Amadei [2] and Pan [3] have successfully applied the BEM to the analysis of anisotropic solids with cracks under the mechanical load.

Nevertheless, when the thermal effects are considered, the extra volume integral terms arise in the integral equations, which negate the advantages of the BEM. For the case of isotropic solids, volume integrals can be transformed to the boundary ones. Prasad et al. [4], Mukherjee et al. [5], Koshelev and Ghassemi [6] utilized this transformation approach in the BEM for studying of cracked isotropic thermoelastic solids. However, in the case of anisotropic solids the transformation of the volume integral to the boundary one is a complicated problem. Tokovyy and Ma [7] obtained the Volterra integral equations of thermoelasticity for the orthotropic plane, half-plane and a strip. For a solid of an arbitrary shape, several methods have been used. Deb and Banerjee [8] and Deb et al. [9] developed the particular integral approach, which involves subdividing of the domain occupied by a solid and carrying out multiple regression analyses to approximate the temperature field in each of the sub-domains as simple polynomials. Shiah and Tan [10] and Shiah et al. [11] proposed the algorithm of exact volume-to-surface integral transformation. Shiah and Tan [12] successfully applied this approach to the analysis of cracked thermoelastic anisotropic solids. However, some of the boundary integrals are evaluated in the mapped temperature domain, which complicates the BEM algorithm. Qin et al. [13–16] obtained the BEM for anisotropic pyroelectric solids with thermally insulated cracks, holes and inclusions minimizing the related potential energy and using Green's function method. However, the latter approach does not consider cracks with temperature boundary conditions set on them.

This paper derives the boundary integral equations for plane problems of thermoelasticity of anisotropic solids of arbitrary shape using the theory of analytic functions [17,18]. This approach has been utilized to obtain the boundary integral equations for anisotropic [19] and piezoelectric [20] solids. Using the obtained new identities of the extended Stroh formalism, this paper develops the Somigliana type integral equations for the 2D anisotropic thermoelasticity.

2. Governing equations and the extended Stroh formalism

Consider a steady state (or quasi-stationary) thermal and mechanical fields acting in a solid placed in a fixed rectangular coordinate system $Ox_1x_2x_3$. The balance of a heat flux and a static equilibrium of the solid are governed by the following

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http://dx.doi.org/10.1016/j.enganabound.2012.07.007

equations [21, 22]:

$$h_{i,i}=0,\quad \sigma_{ij,j}=0, \tag{1}$$

where h_i is a heat flux; σ_{ij} is a stress tensor. According to [21, 22], the constitutive relations of the linear heat conduction (Fourier's law) and thermoelasticity (generalized Hook's law) of anisotropic solids are as follows:

$$h_i=-k_{ij}\theta_{,j},\quad \sigma_{ij}=C_{ijkm}\varepsilon_{km}-\beta_{ij}\theta. \tag{2}$$

Here $\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$ is a strain tensor; u_i is a displacement vector; θ is a temperature change with respect to the reference temperature; C_{ijkm} are elastic moduli; k_{ij} are heat conduction coefficients; $\beta_{ij}=C_{ijkm}\alpha_{km}$ are thermal moduli; α_{ij} are thermal expansion coefficients. Tensors C_{ijkm}, k_{ij}, β_{ij} and α_{ij} are assumed to be symmetric. Here and further, the Einstein summation convention is used. The comma at subscript denotes the differentiation with the respect to the coordinate indexed after the comma, i.e. $u_{i,j}\equiv\partial u_i/\partial x_j$.

Consider a cylindrical solid, in which temperature and displacement fields do not change in the Ox_3 direction, i.e. $\theta_{,3}\equiv 0$, $u_{i,3}\equiv 0$ (plane strain conditions). In this case, one can study only the 2D fields acting at the arbitrary cross-section of the solid normal to the Ox_3 axis.

For the plane problem, Eqs. (1) and (2) for heat conduction are reduced to a homogeneous second-order differential equation

$$k_{ij}\theta_{,ij}=0, \tag{3}$$

which solution can be expressed as [21]

$$\theta=2\mathrm{Re}\{g'(z_t)\},\quad z_t=x_1+p_t x_2. \tag{4}$$

Here $g(z_t)$ is an analytic function of a complex variable z_t; the prime $(\bullet)'$ denotes differentiation with respect to the argument; p_t is a complex constant with positive imaginary part, which is a root of the characteristic equation

$$k_{22}p_t^2+2k_{12}p_t+k_{11}=0. \tag{5}$$

According to Eq. (2), the components h_i of the heat flux vector are equal to the partial derivatives of the heat flux function ϑ [13, 21]

$$h_1=-\vartheta_{,2},\quad h_2=\vartheta_{,1},\quad \vartheta=2k_t\,\mathrm{Im}\{g'(z_t)\},\quad k_t=\sqrt{k_{11}k_{22}-k_{12}{}^2}. \tag{6}$$

The heat flux function ϑ is a scalar function, which equals in magnitude to the heat energy conducted through some arc (actually, a cylindrical surface of a unit width normal to the Ox_1x_2 plane). This can be easily verified by evaluation of the heat energy conducted through an arc ss_0

$$\int_{s_0}^{s}h_in_i\,ds=\int_{s_0}^{s}(h_1\,dx_2-h_2\,dx_1)=\int_{s_0}^{s}(-\vartheta_{,2}\,dx_2-\vartheta_{,1}\,dx_1)=-\vartheta(s)+\vartheta(s_0). \tag{7}$$

Eq. (7) shows that the heat flux function is defined to within a constant.

Eqs. (1) and (2) for thermoelastic equilibrium are reduced to the non-homogeneous second-order differential equations [21, 23]

$$C_{ijkm}u_{k,jm}-\beta_{ij}\theta_{,j}=0, \tag{8}$$

which homogeneous part is as follows:

$$C_{ijkm}u_{k,jm}=0. \tag{9}$$

According to Refs. [21, 23], the general solution of the homogeneous equation (9) is sought in the class of analytic functions $F(x_1+px_2)$ as

$$u_k=a_kF(x_1+px_2). \tag{10}$$

Substituting Eq. (10) into Eq. (9) one can obtain the equation for determination of unknown constants a_k and p:

$$C_{ijkm}(\delta_{j1}+p\delta_{j2})(\delta_{m1}+p\delta_{m2})a_k=0\quad (i,k=1,..,3), \tag{11}$$

where δ_{ij} is a Kronecker delta. In matrix notation, Eq. (11) can be written as

$$[\mathbf{Q}+p(\mathbf{R}+\mathbf{R}^{\mathrm{T}})+p^2\mathbf{T}]\mathbf{a}=\mathbf{0}. \tag{12}$$

Here the components of 3×3 matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are defined as $Q_{ik}=C_{i1k1}$, $T_{ik}=C_{i2k2}$, $R_{ik}=C_{i1k2}=C_{k2i1}$. Matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric due to the symmetry of elastic moduli C_{ijkm}.

Eq. (12) is reduced [23] to the eigenvalue problem for the elasticity matrix $\mathbf{N}$

$$\mathbf{N}=\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}}\end{bmatrix},\quad \mathbf{N}\boldsymbol{\xi}=p\boldsymbol{\xi},\quad \mathbf{N}^{\mathrm{T}}\boldsymbol{\eta}=p\boldsymbol{\eta}, \tag{13}$$

where $\mathbf{N}_1=-\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}$; $\mathbf{N}_2=\mathbf{T}^{-1}$; $\mathbf{N}_3=\mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}-\mathbf{Q}$; $\boldsymbol{\xi}=[\mathbf{a},\mathbf{b}]^{\mathrm{T}}$ is a right eigenvector and $\boldsymbol{\eta}=[\mathbf{b},\mathbf{a}]^{\mathrm{T}}$ is a left eigenvector of the elasticity matrix $\mathbf{N}$; superscript "T" denotes matrix transpose. Vectors $\boldsymbol{\xi}_\alpha$ and $\boldsymbol{\eta}_\beta$ obtained for the eigenvalues p_α and p_β are normalized as follows:

$$\boldsymbol{\xi}_\alpha^{\mathrm{T}}\boldsymbol{\eta}_\beta=\delta_{\alpha\beta}. \tag{14}$$

Eigenvalues p_α (and corresponding vectors $\mathbf{a}_\alpha$) cannot be real [23], therefore, the problem (13) produces six complex eigenvalues p_α and $p_{\alpha+3}=\overline{p}_\alpha$ $(\alpha=1,..,3)$. Corresponding eigenvectors $\boldsymbol{\xi}_\alpha=\overline{\boldsymbol{\xi}}_{\alpha+3}$ are also complex conjugate. Thus, the general solution of Eq. (9) is obtained by superposing six solutions of the form (10) associated with six eigenvalues p_α.

The particular solution of the non-homogeneous equation (8) is written as [21, 23]

$$u_i=c_ig(z_t). \tag{15}$$

Substituting Eq. (15) into Eq. (8) one can obtain the following matrix equation for determination of the vector $\mathbf{c}$:

$$[\mathbf{Q}+p_t(\mathbf{R}+\mathbf{R}^{\mathrm{T}})+p_t\mathbf{T}]\mathbf{c}=\boldsymbol{\beta}_1+p_t\boldsymbol{\beta}_2, \tag{16}$$

where $\boldsymbol{\beta}_1=[\beta_{i1}]$ and $\boldsymbol{\beta}_2=[\beta_{i2}]$.

Since the displacement is a real function, the general solution of Eq. (8) can be written as a real (or imaginary) part of the sum of homogeneous solutions (10) associated with complex eigenvalues p_α and $p_{\alpha+3}=\overline{p}_\alpha$ $(\alpha=1,..,3)$, and a particular solution (15) [21, 23]

$$\mathbf{u}=2\mathrm{Re}\{\mathbf{A}\mathbf{f}(z_*)+\mathbf{c}g(z_t)\}, \tag{17}$$

where $\mathbf{A}\equiv[A_{i\alpha}]=[\mathbf{a}_\alpha]$; $\mathbf{a}_\alpha$ is an eigenvector of Eq. (12) that corresponds to the eigenvalue p_α $(\alpha=1,2,3)$; $\mathbf{f}(z_*)=[F_1(z_1),F_2(z_2),F_3(z_3)]^{\mathrm{T}}$; $z_\alpha=x_1+p_\alpha x_2$.

After differentiation of Eq. (17), using constitutive relations (2) one can obtain stresses at the arbitrary point through Eqs. [21, 23]

$$\sigma_{i1}=-\varphi_{i,2},\quad \sigma_{i2}=\varphi_{i,1}\quad (i=1,..,3). \tag{18}$$

Here

$$\boldsymbol{\varphi}=2\mathrm{Re}\{\mathbf{B}\mathbf{f}(z_*)+\mathbf{d}g(z_t)\} \tag{19}$$

is a stress function. The matrix $\mathbf{B}\equiv[B_{i\alpha}]=[\mathbf{b}_\alpha]$ and a vector $\mathbf{d}$ are defined as follows [21, 23]:

$$\mathbf{b}_\alpha=(\mathbf{R}^{\mathrm{T}}+p_\alpha\mathbf{T})\mathbf{a}_\alpha=-(\mathbf{Q}+p_\alpha\mathbf{R})\frac{\mathbf{a}_\alpha}{p_\alpha}, \tag{20}$$

$$\mathbf{d}=(\mathbf{R}^{\mathrm{T}}+p_t\mathbf{T})\mathbf{c}-\boldsymbol{\beta}_2=-\frac{1}{p_t}(\mathbf{Q}+p_t\mathbf{R})\mathbf{c}+\frac{1}{p_t}\boldsymbol{\beta}_1. \tag{21}$$

The Stroh orthogonality relations (14) can be written through the matrices $\mathbf{A}$ and $\mathbf{B}$ in the following form [23]:

$$\begin{bmatrix}\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}\\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \overline{\mathbf{A}}\\ \mathbf{B} & \overline{\mathbf{B}}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}. \tag{22}$$