Engineering Analysis with Boundary Elements 36 (2012) 1618–1630

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

On the superiority of the mixed element free Galerkin method for solving the steady incompressible flow problems

Xiaodong Wang, Jie Ouyang*, Jin Su, Binxin Yang

Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710129, China

ARTICLE INFO

Article history:
Received 14 November 2011
Accepted 10 May 2012
Available online 20 June 2012

Keywords:
Meshfree
Mixed EFG
Locking
Inf–sup
Incompressible
Steady flow

ABSTRACT

The element free Galerkin (EFG) method is a promising method for solving flow problems, but it meets the difficulty of volumetric locking for solving the incompressible flow problems. In this paper, a mixed EFG method is proposed for solving the steady incompressible flow problems, which avoids the volumetric locking and inherits the meshfree properties. The method employs two sets of nodes, one for the velocity approximation and the other for the pressure approximation. Specially, the ratio between the velocity node number and the pressure node number is taken as the only indicator for the locking behavior of the mixed EFG method. And inf–sup tests are carried out to investigate the relationship between the ratio and locking behavior. By two numerical examples, the accuracy, rate of convergence and efficiency of the mixed EFG method are also carefully studied. The results show that the accuracy, convergence and efficiency of the mixed EFG method are superior to that of the time-related fractional step methods.

1. Introduction

Recently, a large variety of meshfree methods [1–7] have been proposed for flow problems. Element free Galerkin (EFG) method [8], which uses the moving least squares (MLS) approximation to construct shape functions and Galerkin weak-form to formulate partial differential equations, is one of the most famous meshfree methods.

For the unsteady flow problems, several algorithms based on the EFG method have been successfully developed. Qiu et al. [9] combined the EFG method with a fractional step method [10,11] to simulate flow around a cylinder. Li and Duan [12] and Zhang et al. [13] introduced the Characteristic-Based Split (CBS) scheme to the EFG method for incompressible Navier–Stokes equations. Xiong and Wang [14] combined the EFG method with the SIMPLER algorithm [15] to solve incompressible viscous flow problems. When it is to the steady flow problems, these methods are often used by means of relating the problems with time and getting approximate solutions through time-marching. However, the efficiency of these methods for steady flow problems is very low because too many iteration steps are needed. Furthermore, their accuracy is reduced by operator splitting.

An alternate way for solving the steady flow problems uses the EFG method in the mixed form [16,17]. However, this way often encounters the problem of volumetric locking [18,19] in the case of incompressible or nearly incompressible [20,21]. In Refs. [8,22], it was claimed that the EFG method does not exhibit volumetric locking when the shape functions were constructed with large domains of influence. However, large domains of influence not only increase the computation cost but also decrease the accuracy.

In this study, a mixed EFG method is proposed for solving the steady incompressible flow problems to avoid volumetric locking by means of using different approximations for the velocity and the pressure. The method uses two sets of nodes, one for the velocity and the other for the pressure. The two sets of nodes can be generated independently. The velocity and the pressure are approximated by the MLS method based on their own set of nodes, and the approximation orders may be different. The interpolations of the solution values from one set of nodes to the other are performed by the MLS approximations. Volumetric locking can be avoided using this mixed EFG method only if the numbers of velocity nodes are more than that of pressure nodes. Compared with the time-related methods, the mixed EFG method solves the flow variables without relating the steady Navier–Stokes equations with time and decoupling them, so that huge computational cost caused by time-marching and errors caused by splitting algorithms can be removed.

The outline of this paper is as follows: Section 2 is the basic formula of the mixed EFG method for steady flow problems, including shape function construction, mixed variational formulation, inf–sup condition and mixed modes. In Section 3, the locking-free behavior of the mixed EFG method is tested and verified. Then, the accuracy and efficiency of the mixed EFG method are carefully studied in Sections 4 and 5, respectively. Section 6

* Corresponding author. Tel.: +86 29 8849 5234; fax: +86 29 8849 1000.
E-mail address: jieouyang@nwpu.edu.cn (J. Ouyang).

0955-7997/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.enganabound.2012.05.006

investigates the performance of the mixed EFG method on the numerical example of lid-driven cavity flow. Finally, some brief conclusions are drawn in Section 7.

2. Basic formulas

2.1. Steady Navier–Stokes equations

Steady flows of viscous incompressible fluids are governed by the steady Navier–Stokes equations. The strong form of the boundary value problem is stated as follows: given the body force $\mathbf{b}$ in domain Ω, velocities $\overline{\mathbf{u}}$ on boundary portion Γ_D and tractions $\mathbf{t}$ on the remaining portion Γ_N, determine the velocity field $\mathbf{u}$ and the pressure field p, such that

$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega, \tag{1}$$

$$-\frac{1}{Re}\nabla^2\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{b} \quad \text{in } \Omega, \tag{2}$$

along with the boundary conditions

$$\mathbf{u}=\overline{\mathbf{u}} \quad \text{on } \Gamma_D \tag{3}$$

$$-p\mathbf{n}+\frac{1}{Re}(\mathbf{n}\cdot\nabla)\mathbf{u}=\mathbf{t} \quad \text{on } \Gamma_N, \tag{4}$$

where Re is the Reynolds number defined as $Re=VL/\nu$. V, L and ν represent the characteristic velocity, characteristic length and kinematic viscosity, respectively.

2.2. MLS approximation

Consider an unknown scalar function $u(\mathbf{x})$ in the domain Ω. The MLS approximation of $u(\mathbf{x})$ at $\mathbf{x}$ is defined as

$$u^h(\mathbf{x})=\sum_{j=1}^{m}p_j(\mathbf{x})a_j(\mathbf{x})=\mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}), \tag{5}$$

where $\mathbf{p}^T(\mathbf{x})$ is the basis vector of the spatial coordinates, $\mathbf{x}=(x_1,x_2)^T$ for two dimensional problem, and m is the number of the basis functions. In two dimensional cases, the basis function $\mathbf{p}^T(\mathbf{x})$ is often given by

Constant basis $\mathbf{p}^T=(1)$,
Linear basis $\mathbf{p}^T=(1,x_1,x_2)$,
Quadratic basis $\mathbf{p}^T=(1,x_1,x_2,x_1^2,x_1x_2,x_2^2)$.

In Eq. (5), $\mathbf{a}(\mathbf{x})$ is the vector of coefficients given by

$$\mathbf{a}(\mathbf{x})=(a_1(\mathbf{x}),a_2(\mathbf{x}),\ldots,a_m(\mathbf{x}))^T \tag{6}$$

To determine the coefficient vector $\mathbf{a}(\mathbf{x})$, we define a weighted discrete L_2 norm as follows:

$$R(\mathbf{x})=\sum_{i=1}^{n}w(\mathbf{x}-\mathbf{x}_i)[\mathbf{p}^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x})-u_i]^2, \tag{7}$$

where n, $w(\mathbf{x}-\mathbf{x}_i)$, and $u_i=u(\mathbf{x}_i)$ represent the node number in the support domain of $\mathbf{x}$, node weight functions and values of the function at node $\mathbf{x}_i$, respectively. In this paper, we choose the cubic spline as the weight functions, i.e.

$$w(\mathbf{x}-\mathbf{x}_I)=w(r)=\begin{cases}2/3-4r^2+4r^3 & r\le 1/2\\ 4/3-4r+4r^2-4r^3/3 & 1/2<r\le 1\\ 0 & r>1\end{cases} \tag{8}$$

In order to guarantee $u^h(\mathbf{x})$ is the best approximation, we need to minimize $R(\mathbf{x})$, that is

$$\frac{\partial R}{\partial \mathbf{a}}=\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x})-\mathbf{B}(\mathbf{x})\mathbf{U}=\mathbf{0} \tag{9}$$

where

$$\mathbf{A}(\mathbf{x})=\sum_{i=1}^{n}w(\mathbf{x}-\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i), \tag{10}$$

$$\mathbf{B}(\mathbf{x})=(w(\mathbf{x}-\mathbf{x}_1)\mathbf{p}(\mathbf{x}_1),w(\mathbf{x}-\mathbf{x}_2)\mathbf{p}(\mathbf{x}_2),\ldots,w(\mathbf{x}-\mathbf{x}_n)\mathbf{p}(\mathbf{x}_n)) \tag{11}$$

and $\mathbf{U}$ is the vector that collects the values of field function for all the nodes in the support domain.

$$\mathbf{U}=(u_1,u_2,\ldots,u_n)^T \tag{12}$$

Solve $\mathbf{a}(\mathbf{x})$ from (9) and insert it into (5), we get the approximate function of $u(\mathbf{x})$, so that

$$u^h(\mathbf{x})=\mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{U}=\mathbf{\Phi}^T(\mathbf{x})\mathbf{U} \tag{13}$$

where

$$\mathbf{\Phi}^T(\mathbf{x})=(\phi_1(x),\phi_1(x),\ldots,\phi_n(x))^T \tag{14}$$

Notice that the MLS shape functions given in (13) do not, in general, satisfy the Kronecker delta condition. Special techniques, such as the penalty method and Lagrange multiplier method [23], should be used to impose the essential boundary conditions for the EFG method. Because of its simplicity, the penalty method is used in this paper.

2.3. Mixed variational formulation

The variational formulation of the steady Navier–Stokes equations requires local approximations for the velocity components and pressure, as well as for their associated test functions. We denote by $\mathcal{S}^h$ and $\mathcal{V}^h$ the finite dimensional spaces of trial solutions and test functions with respect to the velocity, and $\mathcal{Q}^h$ the finite dimensional space with respect to the pressure. The mixed variational formulation [24] for the steady Navier–Stokes equations then may be stated as follows: for the given $\mathbf{b}$, $\overline{\mathbf{u}}$ and $\mathbf{t}$, find the velocity $\mathbf{u}^h\in\mathcal{S}^h$ and the pressure $p^h\in\mathcal{Q}^h$ for all$(\mathbf{w}^h,q^h)\in\mathcal{V}^h\times\mathcal{Q}^h$, such that

$$\begin{cases}a(\mathbf{w}^h,\mathbf{u}^h)+c(\mathbf{u}^h;\mathbf{w}^h,\mathbf{u}^h)+b(\mathbf{w}^h,p^h)=(\mathbf{w}^h,\mathbf{b}^h)+(\mathbf{w}^h,\mathbf{t}^h)_{\Gamma_N} & \forall\mathbf{w}^h\in\mathbf{V}^h\\ b(\mathbf{u}^h,q^h)=0 & \forall q^h\in\mathcal{Q}^h\end{cases} \tag{15}$$

where

$$a(\mathbf{w}^h,\mathbf{u}^h)=\frac{1}{Re}\int_\Omega\nabla\mathbf{w}^h:\nabla\mathbf{u}^h\,d\Omega, \tag{16}$$

$$c(\mathbf{u}^h;\mathbf{w}^h,\mathbf{u}^h)=\int_\Omega\mathbf{w}^h\cdot(\mathbf{u}^h\cdot\nabla)\mathbf{u}^h\,d\Omega, \tag{17}$$

$$b(\mathbf{w}^h,p^h)=-\int_\Omega p^h\overline{\nabla}\cdot\mathbf{w}^h\,d\Omega, \tag{18}$$

$$b(\mathbf{u}^h,q^h)=-\int_\Omega q^h\nabla\cdot\mathbf{u}^h\,d\Omega, \tag{19}$$

$$(\mathbf{w}^h,\mathbf{b}^h)=\int_\Omega\mathbf{w}^h\mathbf{b}^h\,d\Omega, \tag{20}$$

$$(\mathbf{w}^h,\mathbf{t}^h)_{\Gamma_N}=\int_{\Gamma_N}\mathbf{w}^h\mathbf{t}^h\,d\Gamma, \tag{21}$$

and

$$\overline{\nabla}\cdot\mathbf{w}^h=(\partial w_1^h/\partial x_1,\ldots,\partial w_{n_{sd}}^h/\partial x_{n_{sd}})^T, \tag{22}$$

$$\mathbf{w}^h\mathbf{b}^h=(w_1^hb_1^h,\ldots,w_{n_{sd}}^hb_{n_{sd}}^h)^T, \tag{23}$$

with $n_{sd}=2$ for two dimensions and $n_{sd}=3$ for three dimensions. Together with the penalty method for essential boundary conditions, (15) becomes to