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Assessment of global and local meshless methods based on collocation with radial basis functions for parabolic partial differential equations in three dimensions

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ABSTRACT

A comparison of the performance of the global and the local radial basis function collocation meshless methods for three dimensional parabolic partial differential equations is performed in the present paper. The methods are structured with multiquadrics radial basis functions. The time-stepping is performed in a fully explicit, fully implicit and Crank–Nicolson ways. Uniform and non-uniform node arrangements have been used. A three-dimensional diffusion–reaction equation is used for testing with the Dirichlet and mixed Dirichlet–Neumann boundary conditions. The global methods result in discretization matrices with the number of unknowns equal to the number of the nodes. The local methods are in the present paper based on seven-noded influence domains, and reduce to discretization matrices with seven unknowns for each node in case of the explicit methods or a sparse matrix with the dimension of the number of the nodes and seven non-zero row entries in case of the implicit method. The performance of the methods is assessed in terms of accuracy and efficiency. The outcome of the comparison is as follows. The local methods show superior efficiency and accuracy, especially for the problems with Dirichlet boundary conditions. Global methods are efficient and accurate only in cases with small amount of nodes. For large amount of nodes, they become inefficient and run into ill-conditioning problems. Local explicit method is very accurate, however, sensitive to the node position distribution, and becomes sensitive to the shape parameter of the radial basis functions when the mixed boundary conditions are used. Performance of the local implicit method is comparatively better than the others when a larger number of nodes and mixed boundary conditions are used. The paper represents an extension of our recently made similar study in two dimensions.

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1. Introduction

In recent years radial basis functions (RBFs) have been extensively used in different applications [2,3,5,17,32,36,37,39,43] and emerged as a potential alternative in the field of numerical solution of partial differential equations (PDEs). A detailed discussions on meshless methods and their applications to many complex PDEs, industrial and large-scale problems can also be found in [8,11,12,22–24,40,42] and the references therein.

Different types of meshless methods, based on RBFs, have gained popularity in the engineering and science community for a number of reasons. The most attractive features of the meshless methods are (i) they provide an alternative numerical tool, free from extensive and costly mesh generation or manipulation related problems; (ii) they are flexible in dealing with complex geometries, and are easily extendible to multi-dimensional problems. Meshless methods have been proved successful for solving PDEs on both regular and irregular node arrangements. They use functional basis which allows arbitrary placement of points. Traditional numerical methods, such as the finite difference method (FDM), the finite volume method (FVM), and the finite element method (FEM), are based on the local mesh based interpolation to find the solution and its derivatives. In contrary to these mesh-based methods, meshless methods use a set of uniform or random points which are not interconnected in the form of a classical mesh. Meshless methods actually reduce to multivariate data fitting between the points and related calculation of the derivatives and/or integrals. In the case of meshless methods, interpolation can be accomplished both locally and globally with high efficiency.

In 1971, Hardy introduced radial basis functions interpolation [13] to approximate two-dimensional geographical surfaces based on scattered data. Later on, meshless methods, based on

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Multiquadric (MQ) RBFs [15], were derived for numerical solutions of different types of PDEs. The idea was extended by [10] afterwards. The existence, uniqueness, and convergence of the RBFs approximation was discussed in [9,26,28]. The importance of shape parameter c in the MQ RBF was elaborated in [35]. Solvability of the system of equations with respect to distinct interpolation points was discussed in [28]. All of these methods can be called the global radial basis function collocation method (GRBFCM). The most recent applications of the GRBFCM can be found in [1,17,32–34]. The main disadvantage of the GRBFCM is that it involves full matrices that result from the discretization of the PDEs. These matrices are often ill-conditioned and extremely sensitive to the choice of the shape parameters in RBFs.

To overcome the problems of ill-conditioning and shape parameter sensitivity of the GRBFCM, the local radial basis function collocation method (LRBFCM) was first introduced for diffusion problems in [43] with improved results in terms of accuracy and efficiency of the method. Subsequently, due to handiness of this approach, the LRBFCM has been applied to more complex problems such as convection–diffusion problems with phase-change [37], continuous casting [40], solid–solid phase transformations [22], heat transfer and fluid flow [41], Navier–Stokes equations [7], Darcy flow [19], turbulent flow [39], etc.

The main idea of LRBFCM is the collocation on the overlapping sub-domains of influence instead of the whole domain which drastically reduces the size of the collocation matrix at the expense of solving many small matrices. The size of each small matrix is the same as the number of nodes included in the domain of influence of each node.

The main disadvantage of the LRBFCM is that the method does not work for elliptic problems in a straightforward way. Another kind of RBF-based meshless methods use the integration of RBFs instead of the differentiation of the RBFs. They are known as indirect RBF collocation methods. This class includes indirect RBFN method (IRBFN) [27], the method of approximate particular solutions (MAPS) [6], the localized method of approximate particular solutions (LMAPS) [47], and others. The recent studies can be found in [49]. The LMAPS works well for elliptic PDEs, and can be extended to time-dependent problems as well [46]. This approach yields sparse matrices instead of full matrices, which makes the LMAPS suitable for solving large-scale problems. However, in this paper, we will focus on the LRBFCM only.

PDEs govern physical problems like transport processes, including heat transfer and fluid flows, wave propagation or interaction between fluids and solids, and option pricing. Unlike lower-dimensional problems, the numerical simulation of three-dimensional problems [4,44,45] is much more computationally intensive in terms of CPU time and huge memory requirements. Local meshless methods are not that much prone to these problems since the coefficient matrix is of the same size as the size of the local sub-domain, which is usually relatively small. In the case of uniform node arrangement, the small matrix needs to be inverted only once outside the time-loop for time-dependent problems. This saves a considerable amount of CPU time and consumes less memory as well. The computational efficiency of the local meshless methods in the case of two-dimensional problems and its usefulness in large-scale simulations can be found in [19–21,25,38,39,48].

This paper is an extension of work in [48] to three-dimensional problems. The main motivation for this work is that literature on the numerical methods for three-dimensional problems is sparse compared with lower-dimensional problems. This is particularly true in the field of meshless methods. Some of the related work can be found in Refs. [4,14,18,44,45]. We compare performances of the following five meshless collocation methods: the global implicit radial basis function collocation method (GI), the global Crank–Nicolson radial basis function collocation method (GCN), the global explicit radial basis function collocation method (GE), the local explicit radial basis function collocation method (LE), and the local implicit radial basis function collocation method (LI).

The structure of the rest of the paper is as follows. In Section 2, we introduce the governing equations. In Section 3, we discuss the time discretization technique from implicit and explicit points of view. In Section 4, the numerical methods are introduced from the local and global views. Section 5 is devoted to discussion regarding the scaling technique of the shape parameter c of MQ RBF and the numerical tests on benchmark problems. At the end, conclusions are drawn.

2. Governing equations

Consider a dimensionless form of the three-dimensional diffusion–reaction equation, defined on domain Ω with boundary Γ

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \mathcal{L}[u(\mathbf{x},t)] + \mu u(\mathbf{x},t) + g(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \ t > t_0, \tag{1}$$

with the initial condition

$$u(\mathbf{x},t_0) = u_0, \quad \mathbf{x} \in \Omega \cup \Gamma, \tag{2}$$

and Dirichlet or Neumann boundary conditions

$$\mathcal{B}[u(\mathbf{x},t)] = f(\mathbf{x},t), \quad t \geq t_0, \ \mathbf{x} \in \Gamma, \tag{3}$$

where $u,t,\mathbf{x} = [x,y,z]^{tr}$ are the diffusion, time and space variables, respectively, tr represents the matrix transpose, g and f are the known functions of $\mathbf{x}$ and t, $\Gamma = \Gamma_D + \Gamma_N$, where Γ_N and Γ_D are the boundaries that satisfy Neumann and Dirichlet boundary conditions, respectively. μ is a real constant, $\mathcal{L}$ is a differential operator consisting of first- or second-order derivatives of space variables and $\mathcal{B}$ is a first-order differential operator with respect to space variables in the case of the Neumann boundary conditions and is identity operator in the case of the Dirichlet boundary conditions.

3. Time discretization

Let Δt be the time-step size, and $t = t_0 + \Delta t$ be the time discretization, where t_0 refers to the beginning time of every time step, and t refers to the end of the time step. For a time period $[t_0,t]$, the time derivative in Eq. (1) is approximated by Euler formula:

$$\frac{\partial u(\mathbf{x},t)}{\partial t} \approx \frac{u(\mathbf{x},t) - u(\mathbf{x},t_0)}{\Delta t}. \tag{4}$$

Let $\theta \in (0,1]$. The parameter θ is used in the time discretization of Eq. (1) as

$$u(\mathbf{x},t_0 + \theta\Delta t) \approx \theta u(\mathbf{x},t) + (1-\theta)u(\mathbf{x},t_0), \tag{5}$$

$$g(\mathbf{x},t_0 + \theta\Delta t) \approx \theta g(\mathbf{x},t) + (1-\theta)g(\mathbf{x},t_0), \tag{6}$$

$$\mathcal{L}[u(\mathbf{x},t_0 + \theta\Delta t)] \approx \theta\mathcal{L}[u(\mathbf{x},t)] + (1-\theta)\mathcal{L}[u(\mathbf{x},t_0)]. \tag{7}$$

Then Eq. (1) can be discretized in time-space as

$$\begin{aligned}(1-\mu\theta\Delta t)u(\mathbf{x},t) - \theta\Delta t\mathcal{L}[u(\mathbf{x},t)] - \theta\Delta t g(\mathbf{x},t) \\ = (1+\mu(1-\theta)\Delta t)u(\mathbf{x},t_0) + (1-\theta)\Delta t\mathcal{L}[u(\mathbf{x},t_0)] \\ + (1-\theta)\Delta t g(\mathbf{x},t_0),\end{aligned} \tag{8}$$

note that $t = t_0 + \Delta t$. Similarly, for $\mathbf{x} \in \Gamma$, Eq. (3) can be discretized in time-space as

$$\theta\mathcal{B}[u(\mathbf{x},t)] - \theta f(\mathbf{x},t) = -(1-\theta)\mathcal{B}[u(\mathbf{x},t_0)] + (1-\theta)f(\mathbf{x},t_0). \tag{9}$$

To represent the approximate solution of Eqs. (1)–(3) in a single equation, at the interior and boundary points, we define the following