Engineering Analysis with Boundary Elements 44 (2014) 55–63

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

# A fractional step local boundary integral element method for unsteady two-dimensional incompressible flow

Karel Kovářík*, Juraj Mužík, Dana Sitányiová

*University of Žilina, Faculty of Civil Engineering, Department of Geotechnics, Univerzitná 8215/1, 010 26 Žilina, Slovakia*

ARTICLE INFO

*Article history:*
Received 7 May 2013
Received in revised form
17 February 2014
Accepted 10 April 2014
Available online 9 May 2014

*Keywords:*
LBIEM
Incompressible fluid flow
Meshless method
Radial basis functions

ABSTRACT

A meshless local boundary integral method is used to analyse incompressible fluid flow in two-dimensional domains. The method solves the incompressible Navier–Stokes equations in terms of the primitive variables using the characteristic-based split scheme. The basic equations are derived via interpolation using radial basis functions. Three test cases are presented here: unsteady Couette flow and the problems of a lid-driven cavity and a backward-facing step. The procedure produces stable solutions with results comparable to those of other conventional methods.



## 1. Introduction

The solving of incompressible Navier–Stokes equations together with the continuity equation constitutes a major problem of computational fluid dynamics. The finite element and volume methods are usually used to solve this problem. However, these equations are nonlinear, which leads to the simulation usually requiring large meshes and lengthy computation. The problems associated with the non-linearity of the Navier–Stokes equations can be overcome using a characteristic-based splitting (CBS) scheme, which was developed to solve this problem in primitive variables (e.g. [1–3]).

Meshless methods have recently been demonstrated as attractive tools for solving computational fluid dynamics problems. They do not require the mesh generation employed by other methods, because they are defined on a cloud of points. Therefore, they are less computationally expensive than mesh-generation methods. The problems of mesh refinement can also be avoided without a notable increase of computational burden. These features make meshless methods powerful tools in the solving of a wide range of scientific problems. During the past year, various meshless methods have been developed by different groups. These include smooth particle hydrodynamics (SPH) [4], the least square collocation meshless method [5–7], the meshless local Petrov–Galerkin (MLPG) method [8,9], the local boundary integral element method (LBIEM) [10–12], and the radial basis integral equation method (RBIEM) [13–15]. SPH and the least square collocation method are strong methods and require no integration, but they show deficiencies in the formulation of boundary conditions, and their solutions are highly sensitive to the selection of local collocation points. The MLPG method, the LBIEM, and the RBIEM are local weak methods, and they can easily deal with different boundary conditions.

The MLPG method was first applied to solve the incompressible flow equations by Lin and Atluri [9], who reported the use of the direct solution of a system of non-linear equations together with an upwind scheme to overcome oscillations produced by the convection term. They also added the perturbation term to the continuity equation to satisfy the Babuska–Brezzi condition. However, the problem remains of finding a proper value of the perturbation parameter in cases of flow with higher Reynolds numbers. Meshless methods can also be aided by the use of CBS to overcome the problem of the non-linearity of the Navier–Stokes equations in primitive variables (e.g., [16]).

The LBIEM introduced by Zhu et al. [10], like other meshless methods, is based on regularly or randomly distributed nodal points covering the domain. Every node is at the centre of a surrounding circular mesh of boundary elements. The unknown variable in this point is then expressed using a boundary integral equation on this local mesh. All of these unknown variables are approximated by some interpolation method to obtain a system of linear equations. Solving this system of equations leads to a numerical solution of the problem. Zhu et al. [10] originally used the moving least squares method for interpolation, but interpolation can now be conducted using radial basis functions (RBFs) (e.g., [17,12]). The LBIEM was applied by Sellountos and Sequeira [17] to solve the incompressible flow using the velocity–vorticity formulation. Approaches using the vorticity–stream or the velocity–vorticity

* Corresponding author. Tel.: +421 905 659763.
*E-mail address:* kovarik@fstav.uniza.sk (K. Kovářík).

http://dx.doi.org/10.1016/j.enganabound.2014.04.010
0955-7997/

formulation cannot be directly extended to solve in 3D, and some types of boundary conditions are also difficult to define. Therefore, the primitive variables method is applied here.

The RBIEM is also based on the boundary integral equations formulated on the local mesh of elements at every point. Unlike the LBIEM, it does not use the companion solution and does not need any integration about the global boundary. The solution of integrals in the RBIEM is simpler than that in the LBIEM, but the RBIEM requires more equations than does the LBIEM. Bui and Popov [13] used the RBIEM to solve Navier–Stokes equations directly, but this formulation required six non-linear equations for each point. This led us to focus here on the LBIEM.

The present paper focuses on the LBIEM using primitive variables and the CBS to solve for incompressible viscous flow. Variables are interpolated using RBFs, rather than moving least squares, because they possess the delta property that simplifies the prescription of the essential boundary conditions.

## 2. Governing equations

An unsteady incompressible flow is governed by Navier–Stokes equations, which can be written in their primitive variables as

$$\frac{\partial u_i}{\partial x_i}=0$$
$$\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}-\frac{1}{\rho}\frac{\partial p}{\partial x_i}-\frac{\partial}{\partial x_j}(u_j u_i)-\frac{\partial u_i}{\partial t}=f_i \tag{1}$$

where $u_i$ is the velocity vector component in the direction $i$, $p$ is the pressure, $\nu$ is the kinematic viscosity, $\rho$ is the density of a liquid, and $f_i$ represents body forces component in the direction $i$. A CBS algorithm is used to solve this problem (see [1,16]). The time derivative of the velocity vector in a momentum equation (1) can be replaced with a difference and the following relation is obtained:

$$u_i^{n+1}=u_i^n+\Delta t\left[\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}-f_i-\frac{\partial}{\partial x_j}(u_j u_i)+\frac{\Delta t}{2}u_k\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}(u_j u_i)+f_i\right)\right]^n-\frac{\Delta t}{\rho}\frac{\partial p^{n+1}}{\partial x_i} \tag{2}$$

where upper indexes $n$ and $n+1$ indicate time steps, $\nu$ is the kinematic viscosity and $\Delta t=t^{n+1}-t^n$ is the length of the time interval. The last term in square brackets acts as the stabilizing term (see [2]). Eq. (2) is an explicit formula for the convection and viscous terms, and an implicit one for the pressure term. Eq. (2) is simplified using the fractional time step approximation (e.g., [6,3]), which computes the intermediate velocity $\tilde{\mathbf{u}}$ using the simplified momentum equation

$$\tilde{u}_i=u_i^n+\Delta t\left[\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}-f_i-\frac{\partial}{\partial x_j}(u_j u_i)+\frac{\Delta t}{2}u_k\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}(u_j u_i)+f_i\right)\right]^n \tag{3}$$

Comparing (2) and (3) gives

$$u_i^{n+1}=\tilde{u}_i-\frac{\Delta t}{\rho}\frac{\partial p^{n+1}}{\partial x_i} \tag{4}$$

The intermediate velocity components $\tilde{u}_i$ do not satisfy the continuity equation in (1). The velocity components $u_i^{n+1}$ must satisfy the continuity equation, which implies

$$\frac{\partial}{\partial x_i}\left(\tilde{u}_i-\frac{\Delta t}{\rho}\frac{\partial p^{n+1}}{\partial x_i}\right)=0 \tag{5}$$

A pressure Poisson equation results directly from (5)

$$\frac{\partial^2 p^{n+1}}{\partial x_i\partial x_i}=\frac{\rho}{\Delta t}\frac{\partial \tilde{u}_i}{\partial x_i} \tag{6}$$

with Neumann boundary conditions that accrue from (4)

$$\left(\frac{\partial p}{\partial n}\right)^{n+1}=\frac{\rho}{\Delta t}(\tilde{u}_i-u_i^{n+1})n_i \tag{7}$$

where $n_i$ is the outer normal vector component in the direction $i$.

## 3. Meshless local integral formulation

The area of interest $\Omega$ with the boundary $\Gamma$ is covered by points within the area and also on the global boundary (see Fig. 1). Consider a local circular sub-domain $\Omega_s$ with boundary $\Lambda_s$ centred at every point $s$. This sub-domain is regular around all the internal points, but at the points on the global boundary this local boundary consists of a part of the global boundary intersected with the local sub-domain $\Gamma_s$ (see Fig. 2). To express the local boundary integral form of the Navier–Stokes equations using the CBS algorithm in a domain $\Omega_s$, we apply the weighting residual principle to Eqs. (3), (4) and (6). For simplicity we neglect a body force term to obtain the following weak form:

$$\int_{\Omega_s}\tilde{u}_i w^*\,d\Omega=\int_{\Omega_s}u_i^n w^*\,d\Omega-\Delta t\left[\int_{\Omega_s}u_j\frac{\partial u_i}{\partial x_j}w^*\,d\Omega-\nu\int_{\Omega_s}\frac{\partial^2 u_i}{\partial x_j\partial x_j}w^*\,d\Omega-\frac{\Delta t}{2}\int_{\Omega_s}u_k\frac{\partial}{\partial x_k}\left(u_j\frac{\partial u_i}{\partial x_j}\right)w^*\,d\Omega\right]^n \tag{8}$$

$$\int_{\Omega_s}u_i^{n+1}w^*\,d\Omega=\int_{\Omega_s}\tilde{u}_i w^*\,d\Omega-\frac{\Delta t}{\rho}\int_{\Omega_s}\frac{\partial p^{n+1}}{\partial x_i}w^*\,d\Omega \tag{9}$$

$$\int_{\Omega_s}\frac{\partial^2 p}{\partial x_i\partial x_i}w^*\,d\Omega-\frac{\rho}{\Delta t}\int_{\Omega_s}\frac{\partial \tilde{u}_i}{\partial x_i}w^*\,d\Omega=0. \tag{10}$$

Here $w^*$ is a weighting (test) function. If the test function is chosen to be the fundamental solution of the Laplace equation, then after integration by parts twice in (8) and (10) the following integral equations can be obtained (see also [8,10,18]):

$$\int_{\Omega_s}\tilde{u}_i w^*\,d\Omega=\int_{\Omega_s}u_i^n w^*\,d\Omega-\Delta t\left[\int_{\Omega_s}u_j\frac{\partial u_i}{\partial x_j}w^*\,d\Omega+\nu\left(c_s u_{is}+\int_{\Lambda_s\cup\Gamma_s}\frac{\partial w^*}{\partial n}u_i\,d\Gamma-\int_{\Lambda_s\cup\Gamma_s}w^*\frac{\partial u_i}{\partial n}\,d\Gamma\right)-\frac{\Delta t}{2}\int_{\Omega_s}u_k\frac{\partial}{\partial x_k}\left(u_j\frac{\partial u_i}{\partial x_j}\right)w^*\,d\Omega\right]^n \tag{11}$$

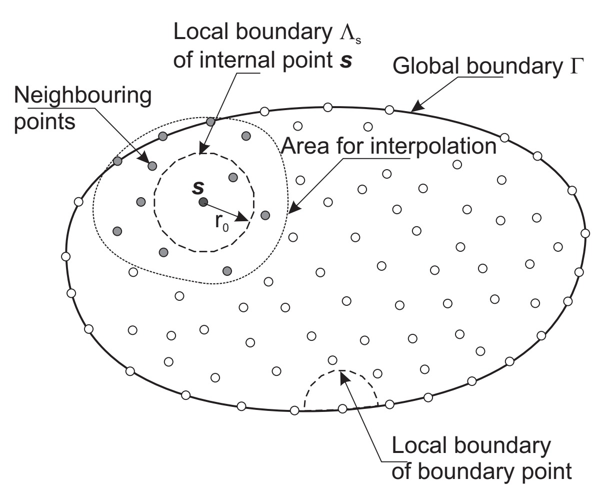


**Fig. 1.** Points in the global area.