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The singular boundary method: Mathematical background and application in orthotropic elastic problems

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ABSTRACT

The singular boundary method (SBM) is a recent strong-form boundary discretization numerical technique and can be viewed as one kind of modified method of fundamental solutions (MFSs). Although the method has been successfully used in many fields of engineering analysis, there has been no attempt yet to present a work discussing the mathematical background of the method. This paper fills this gap in the SBM and documents the first attempt to apply the method to the solution of orthotropic elastic problems. Three benchmark numerical problems are tested to demonstrate the feasibility and accuracy of the proposed method through detailed comparisons with the MFS and the boundary element method (BEM).

1. Introduction

The singular boundary method (SBM) has emerged as an effective meshless boundary collocation method for the solution of certain boundary value problems. This method, based on the notion of boundary element method (BEM) [1–3] and MFS [4–8], fully inherits the merits of both and in the meantime possessing its unique advantages. First, it does not involve integration which could be otherwise troublesome and expensive as in the BEM-based methods. Second, it sidesteps the *fictitious boundary issue* associated with the traditional MFS by means of a concept of the *origin intensity factor*, a numerical strategy that isolates singularities of the fundamental solutions and allows the source and collocation points coincide on the real boundary. The ease of implementation and its low computational cost make the method a competitive alternative for certain boundary value problems. Comprehensive reviews on the SBM for various applications can be found in Refs. [9–12]. Although the method has been successfully used in many fields of engineering analysis, the mathematical background of the method has not, as yet, been discussed.

One of the aims of the present paper is to fill the gap mentioned above and present a relatively detailed theoretical background of this methodology. Related issues in the SBM, such as the existence of the origin intensity factors and their numerical evaluation, will be discussed in detail. This paper also documents the first attempt to extend the method to general orthotropic elastic problems. The study of boundary value problems for orthotropic elastic materials has received considerable attention in recent years [13,14]. This interest is partly related to the extensive use of composite materials in various engineering applications, like wood, reinforced concrete, and all materials that are reinforced with fibers.

A brief outline of the rest of this paper is as follows. In Section 2, we describe the traditional MFS formulation for the solution of 2D orthotropic elastic problems. The SBM formulation and its mathematical background are discussed in details in Section 3. And then in Section 4 the method is tested successfully over three benchmark numerical problems. Finally, the conclusions and remarks are provided in Section 5.

2. MFS formulation for orthotropic elastic problems

For the assumption of plane stress distribution in an orthotropic material, Hooke's law takes the form (matrix representation)

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

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$$= \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{1}$$

where the stress σ_{ij} and strain ε_{ij} are mean values taken through the thickness of the material; $S_{ij}(i,j=1,2)$ and S_{66} are flexibility coefficients; E_1 and E_2 are Young's moduli in the directions of x_1 and x_2 axes; G_{12} denotes the shear modulus for planes parallel to the x_1-x_2 plane; ν_{12} is Poisson's ratio characterizing the contraction in the direction of the x_2 axis when tension is applied in the direction of the x_1 axis.

The Navier–Cauchy equations for plane orthotropic materials, in the absence of body forces, referring to displacements u_1 and u_2 are [15]

$$C_{11}\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_1^2}+(C_{12}+C_{66})\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_1 \partial x_2}+C_{66}\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_2^2}=0, \tag{2a}$$

$$C_{22}\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_2^2}+(C_{12}+C_{66})\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_1 \partial x_2}+C_{66}\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_1^2}=0, \tag{2b}$$

in which $C_{11}=S_{22}/D$, $C_{12}=-S_{12}/D$, $C_{22}=S_{11}/D$, $C_{66}=1/S_{66}$, $D=S_{11}S_{22}-S_{12}^2$.

These are subject to the boundary conditions

$$u_i(\boldsymbol{x})=\overline{u}_i, \quad \boldsymbol{x}\in\Gamma_u(\text{Dirichlet boundary conditions}), \tag{3a}$$

$$t_i(\boldsymbol{x})=\overline{t}_i, \quad \boldsymbol{x}\in\Gamma_t(\text{Neumann boundary conditions}), \tag{3b}$$

where $t_i(\boldsymbol{x})$ denotes the component of boundary traction in the ith coordinate direction, Γ_u and Γ_t construct the whole boundary of the domain Ω which we shall assume to be piecewise smooth, $\overline{u}_i$ and $\overline{t}_i$ represent the prescribed displacements and tractions, respectively.

Employing indicial notation for the coordinates of points $\boldsymbol{x}=(x_1,x_2)$ and $\boldsymbol{y}=(y_1,y_2)$, respectively, the Kelvin fundamental solutions of the systems (2) and (3) can be expressed as [15]

$$U_{11}(\boldsymbol{y},\boldsymbol{x})=K[\sqrt{\alpha_1}A_2^2\ln r_1-\sqrt{\alpha_2}A_1^2\ln r_2] \tag{4a}$$

$$U_{12}(\boldsymbol{y},\boldsymbol{x})=-KA_1A_2\left[\arctan\frac{x_2-y_2}{\sqrt{\alpha_1}(x_1-y_1)}-\arctan\frac{x_2-y_2}{\sqrt{\alpha_2}(x_1-y_1)}\right], \tag{4b}$$

$$U_{21}(\boldsymbol{y},\boldsymbol{x})=U_{12}(\boldsymbol{y},\boldsymbol{x}) \tag{4c}$$

$$U_{22}(\boldsymbol{y},\boldsymbol{x})=K[A_2^2\ln r_2/\sqrt{\alpha_2}-A_1^2\ln r_1/\sqrt{\alpha_1}] \tag{4d}$$

where α_1 and α_2 satisfy

$$\alpha_1+\alpha_2=(2S_{12}+S_{66})/S_{22}, \quad \alpha_1\alpha_2=S_{11}/S_{22},$$

and

$$K=1/[2\pi(\alpha_1-\alpha_2)S_{22}], \quad A_i=S_{12}-\alpha_i S_{22}, \quad r_i=\sqrt{\alpha_i(x_1-y_1)^2+(x_2-y_2)^2}.$$

The fundamental solution $U_{ij}(\boldsymbol{y},\boldsymbol{x})$ described above indicates the displacement produced at point $\boldsymbol{y}$ by a concentrated unit body force applied at point $\boldsymbol{x}$, in which the first subscript (i) denotes the direction of the displacement whereas the second one (j) denotes the direction of the unit force. The fundamental solution of the tractions can be obtained by first calculating the fundamental solutions of strains and then applying Hooke's law

$$T_{11}(\boldsymbol{y},\boldsymbol{x})=K[(x_1-y_1)n_1(\boldsymbol{y})+(x_2-y_2)n_2(\boldsymbol{y})]\left[\frac{A_2\sqrt{\alpha_1}}{r_1^2}-\frac{A_1\sqrt{\alpha_2}}{r_2^2}\right], \tag{5a}$$

$$T_{12}(\boldsymbol{y},\boldsymbol{x})=K\frac{A_2}{r_2^2}[\sqrt{\alpha_2}(x_1-y_1)n_2(\boldsymbol{y})-(x_2-y_2)n_1(\boldsymbol{y})/\sqrt{\alpha_2}] \\ -K\frac{A_1}{r_1^2}[\sqrt{\alpha_1}(x_1-y_1)n_2(\boldsymbol{y})-(x_2-y_2)n_1(\boldsymbol{y})/\sqrt{\alpha_1}], \tag{5b}$$

$$T_{21}(\boldsymbol{y},\boldsymbol{x})=K\frac{\alpha_2A_1}{r_2^2}[\sqrt{\alpha_2}(x_1-y_1)n_2(\boldsymbol{y})-(x_2-y_2)n_1(\boldsymbol{y})/\sqrt{\alpha_2}] \\ -K\frac{\alpha_1A_2}{r_1^2}[\sqrt{\alpha_1}(x_1-y_1)n_2(\boldsymbol{y})-(x_2-y_2)n_1(\boldsymbol{y})/\sqrt{\alpha_1}], \tag{5c}$$

$$T_{22}(\boldsymbol{y},\boldsymbol{x})=K[(x_1-y_1)n_1(\boldsymbol{y})+(x_2-y_2)n_2(\boldsymbol{y})]\left[\frac{A_2\sqrt{\alpha_2}}{r_2^2}-\frac{A_1\sqrt{\alpha_1}}{r_1^2}\right] \tag{5d}$$

By employing the radial basis functions (RBFs) technique [16–19], the displacements and stresses can be approximated by linear combinations of fundamental solutions with respect to different source points $\boldsymbol{x}$ as follows:

$$u_i(\boldsymbol{y}^m)=\sum_{n=1}^{N}a_j(\boldsymbol{x}^n)U_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n)=\sum_{n=1}^{N}[a_1(\boldsymbol{x}^n)U_{i1}(\boldsymbol{y}^m,\boldsymbol{x}^n) \\ +a_2(\boldsymbol{x}^n)U_{i2}(\boldsymbol{y}^m,\boldsymbol{x}^n)], \tag{6a}$$

$$t_i(\boldsymbol{y}^m)=\sum_{n=1}^{N}a_j(\boldsymbol{x}^n)T_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n)=\sum_{n=1}^{N}[a_1(\boldsymbol{x}^n)T_{i1}(\boldsymbol{y}^m,\boldsymbol{x}^n) \\ +a_2(\boldsymbol{x}^n)T_{i2}(\boldsymbol{y}^m,\boldsymbol{x}^n)] \tag{6b}$$

where $i,j=1,2$, N is the specified number of sources, $\boldsymbol{y}^m\in\overline{\Omega}=\Omega\cup\partial\Omega$ is the mth collocation points, $\boldsymbol{x}^n$ is the nth source point, $\{a_1(\boldsymbol{x}^n)\}_{n=1}^N$ and $\{a_2(\boldsymbol{x}^n)\}_{n=1}^N$ denote the unknown coefficients.

In the traditional MFS, a fictitious boundary slightly outside the problem domain is required in order to place the source points and avoid the singularity of the fundamental solutions. These source points are either pre-assigned or taken to be part of the unknowns of the problem along with the coefficients $\{a_j(\boldsymbol{x}^n)\}_{n=1}^N$. In either case, the unknowns are determined so that the approximations (6) satisfy, in some sense, the boundary conditions (3) as close as possible [20,21]. In the early applications of the MFS, the locations of the source points were determined by a non-linear system of the equations that can be solved using a non-linear least-squares minimization software. This approach, however, has attracted limited attention primarily because of its high computational costs and the criticism that a linear boundary value problem is converted to a non-linear discrete problem. In the more established approach these days the source points are pre-assigned, collocation simply leads to a linear system of M equations in N unknowns which can be solved by a least-squares solver. However, despite many years of focused research, the predetermination of the fictitious boundary is largely based on experiences and therefore often troublesome, especially for complicated higher dimensional domain problems [22–27]. This drawback severely downplays the applicability of the MFS to real-world applications.

3. Singular boundary method: mathematical background and numerical implementation

The basic idea of the SBM is to introduce a concept of the *origin intensity factor* to isolate the singularity of the fundamental solutions, so that the source points can be directly placed on the real boundary [10]. With this idea in mind the SBM interpolation can be expressed as

$$u_i(\boldsymbol{y}^m)=\sum_{\substack{n=1\\ m\neq n}}^{N}a_j(\boldsymbol{x}^n)U_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n)+a_j(\boldsymbol{x}^m)A_{ij}(\boldsymbol{x}^m), \tag{7}$$

$$t_i(\boldsymbol{y}^m)=\sum_{\substack{n=1\\ m\neq n}}^{N}a_j(\boldsymbol{x}^n)T_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n)+a_j(\boldsymbol{x}^m)B_{ij}(\boldsymbol{x}^m), \tag{8}$$

where $\boldsymbol{y}^m\in\Gamma$, $A_{ij}(\boldsymbol{x}^m)$ and $B_{ij}(\boldsymbol{x}^m)$ are defined as the origin intensity factors, i.e., the diagonal and sub-diagonal elements of the SBM