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The meshless kernel-based method of lines for the numerical solution of the nonlinear Schrödinger equation

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A R T I C L E I N F O

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A B S T R A C T

In this paper, the nonlinear Schrödinger equation is solved numerically by using the meshless kernel-based method of lines. Multiquadric, Gaussian and Wendland's compactly supported radial basis functions are used as the kernel basis functions. In the numerical examples, the single soliton solution, interaction of two colliding solitons, birth of standing soliton, birth of mobile soliton and bound state of solitons are simulated. The accuracy and efficiency of the used method are tested by computing the lowest two invariants and the relative change of invariants for all test problems. Error norms L_2 and L_∞ are computed for single soliton motion whose exact solution is known. Numerical results and figures off wave motions for all test problems are presented. The numerical solutions of the nonlinear Scrödinger equation are compared with both the analytical solutions and numerical solutions of some earlier papers in the literature.

1. Introduction

The nonlinear Schrödinger (NLS) equation was initially shown to describe the self modulation of a monochromatic wave [1] after it was used in the modelling of several physical phenomena such as the propagation of optical pulses, superconductivity, nonlinear optic, waves in water, waves in plasma and self-focusing effects in laser pulses. The NLS equation has soliton solutions, the solitons are included applications in condensed matter physics, fluid dynamics, elementary particle physics, plasma physics, laser physics, and biophysics.

There is not known theoretical solution of this equation for the general initial conditions. While the initial condition $U(x,0)$ vanishes for sufficiently large x, the analytical solutions of the NLS equation are given in articles [1–6].

The NLS equation was solved exactly in [3] by using the inverse scattering transform for the special cases of solitons. A complex tanh-function method applied to nonlinear equations of Schrödinger type in [4]. An application of He's semi-inverse method to the NLS equation was presented in [5] and reliable analysis for obtaining exact soliton solutions of the NLS equation was presented in [6]. Since theoretical solution of the NLS equation for the general initial conditions is unknown to know the numerical solutions of the NLS equation are essential. Therefore, several numerical methods have been used to find numerical solutions of the NLS equation for different initial conditions [7–24]. These used methods can be summarized as follows:

Finite difference scheme was used in [7,8] to obtain numerical results for the NLS equation. The B-spline finite element methods were used to solve the NLS equation numerically with cubic B-spline finite element and collocation method in [9,10]. Galerkin method with spline functions was used in [11,12]. The orthonormal spline collocation method was applied in articles [13,14]. A quadratic B-spline finite element method was used in [15] for the NLS equation. Local discontinuous Galerkin methods for nonlinear Schrödinger equations were used in [16]. The quadrature algorithms were used in articles [17,18]. The quadrature dicretization method was used in [19,20] for the NLS equation. A comparative analysis of numerical methods to find the numerical solution of the NLS equation by using various finite difference schemes was made in [21]. A conservative scheme was presented in [22] and conservative and nonconservative schemes for the solution of the nonlinear Schrödinger equation were presented in [23]. The meshless radial basis function collocation method was applied to NLS equation in [24].

The aim of this paper is to apply the meshless kernel-based method of lines (MOL) to find numerical results for several test problems of the NLS equation. As kernel functions various radial basis functions which are Multiquadric [25], Gauss and Wendland's [26] compactly supported radial basis functions are used. The accuracy and efficiency of the used method will be demonstrated with numerical examples.

2. The governing equation

The nonlinear Schrödinger equation has the following general form:

$$iU_t + U_{xx} + q|U|^2 U = 0, \quad (x,t) \in [a,b] \times [0,T], \tag{1}$$

$$U(x,0) = g(x), \tag{2}$$

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$$U(a,t)=0, \quad U(b,t)=0, \tag{3}$$

where $i=\sqrt{-1}$, $q \geq 0$ is a real parameter, the subscripts x and $t$denote partial differentiation with respect to space and time, respectively. $U=U(x,t)$ is a complex-valued function.

To solve the NLS equation, we assume that the complex function $U(x,t)$ is decomposed into its real and imaginary parts as follows:

$$U(x,t)=r(x,t)+is(x,t), \tag{4}$$

where $r(x,t)$ and $s(x,t)$ are real functions. Under the this assumption, the complex equation (1) can be written as the coupled pair of real differential equation system:

$$r_t+s_{xx}+q(r^2+s^2)s=0, \tag{5}$$

$$s_t-r_{xx}-q(r^2+s^2)r=0 \tag{6}$$

with initial and boundary conditions as follows:

$$r(x,0)=g_r(x), \quad s(x,0)=g_s(x), \ x \in [a,b], \tag{7}$$

$$r(a,t)=r(b,t)=0, \quad s(a,t)=s(b,t)=0, \ t \in [0,T], \tag{8}$$

where $g_r(x)$ and $g_s(x)$ are, respectively, the real and imaginary parts of $g(x)$.

3. The meshless kernel-based method of lines for the NLS equation

In this section, we present the meshless kernel-based method of lines to obtain the numerical solution of the nonlinear Schrödinger equation (1) with initial and boundary conditions (2) and (3).

The meshless kernel-based method of lines (MOL) is a way of approximating partial differential equations by ordinary differential equations. The meshless kernel-based MOL transforms a partial differential equation to a system of ordinary differential equations, however it does not necessitate the discretization of all variables. The obtained system of ordinary differential equations can be solved by using a standard ordinary differential equations system solver.

Now, let us show application of the meshless kernel-based method of lines to the NLS equation. Generally, the approximate function of $U(x,t)$ is defined by using the meshless kernel based MOL as follows:

$$U(x,t)=\sum_{j=1}^{N}\lambda_j(t)\vartheta_j(x) \tag{9}$$

with smooth functions λ_j on $[0,T]$, $1 \leq j \leq N$ [27]. Here $\lambda_j(t)$ is unknown function as a column vector to be calculated at each time level and $\vartheta_j(x)$ is an invertible matrix for kernel functions. Therefore, using the meshless kernel based MOL approximation given by Eq. (9) approximate values of real functions $r(x,t)$ and $s(x,t)$ are written as follows:

$$r(x,t)=\sum_{j=1}^{N}\alpha_j(t)\upsilon_j(x), \quad s(x,t)=\sum_{j=1}^{N}\beta_j(t)w_j(x) \tag{10}$$

and similarly first and second order derivatives of real functions $r(x,t)$ and $s(x,t)$ are determined. Functions $r(x,t)$, $s(x,t)$ and their derivative functions are substituted into the equation system ((5) and (6)), the following simplified system is obtained:

$$\sum_{j=1}^{N}\alpha_j'(t)\upsilon_j(x)=-\sum_{j=1}^{N}\beta_j(t)w_j''(x)$$
$$-q\left(\left(\sum_{j=1}^{N}\alpha_j(t)\upsilon_j(x)\right)^2+\left(\sum_{j=1}^{N}\beta_j(t)w_j(x)\right)^2\right)\sum_{j=1}^{N}\beta_j(t)w_j(x), \tag{11}$$

$$\sum_{j=1}^{N}\beta_j'(t)w_j(x)=\sum_{j=1}^{N}\alpha_j(t)\upsilon_j''(x)$$
$$+q\left(\left(\sum_{j=1}^{N}\alpha_j(t)\upsilon_j(x)\right)^2+\left(\sum_{j=1}^{N}\beta_j(t)w_j(x)\right)^2\right)\sum_{j=1}^{N}\alpha_j(t)\upsilon_j(x). \tag{12}$$

In this study obtained equation system is solved by using the MATLAB ode113 solver which is a variable order Adams–Bashforth–Moulton PECE solver. This solver is effective to solve computationally intensive problems. Eqs. (11)–(12) can be easily written in MATLAB notation as follows:

$$V*\alpha'(t)=-W''*\beta(t)-q((V*\alpha(t)).\hat{}2+(W*\beta(t)).\hat{}2)\cdot*(W*\beta(t)), \tag{13}$$

$$W*\beta'(t)=V''*\alpha(t)+q((V*\alpha(t)).\hat{}2+(W*\beta(t)).\hat{}2).*(V*\alpha(t)), \tag{14}$$

where "$.*$" defines the pointwise product.

Here matrices V, V'', W and W'' which are invertible matrices and vectors $\alpha(t)$, $\alpha'(t)$, $\beta(t)$ and $\beta'(t)$ are defined as follows:

$$V := (\upsilon_j(x_k)),$$

$$V'' := (\upsilon_j''(x_k)),$$

$$W := (w_j(x_k)),$$

$$W'' := (w_j''(x_k)),$$

$$\alpha(t) := (\alpha_1(t),\ldots,\alpha_N(t))^T,$$

$$\alpha'(t) := (\alpha_1'(t),\ldots,\alpha_N'(t))^T,$$

$$\beta(t) := (\beta_1(t),\ldots,\beta_N(t))^T,$$

$$\beta'(t) := (\beta_1'(t),\ldots,\beta_N'(t))^T$$

for $1 \leq k \leq N$ and $1 \leq j \leq N$. Here k and j are row and column indices, respectively.

4. Kernel functions

In our numerical examples, we have used globally supported radial basis functions which are Multiquadric [25] and Gaussian functions and Wendland's [26] compactly supported radial basis functions.

Multiquadric and Gaussian radial basis functions are defined as follows:

Multiquadric(MQ) $\quad \phi(r)=\sqrt{(\varepsilon r)^2+1},$

Gaussian(G) $\quad \phi(r)=\exp(-r^2/\varepsilon^2),$

where $r=|x-x_j|$ is the Euclidean distance between collocation points x and x_j.

MQ and G radial basis functions are globally supported and infinitely differentiable, therefore obtained kernel matrix for these functions is a full interpolation matrix. Also these functions contain the shape parameter ε. To find the optimal value of shape parameter is still an open question. In this paper, we calculated the condition number of the kernel matrix to obtain the optimal value of shape parameter [28].

In our computations Wendland's compactly supported radial basis functions are used as another type of basis functions. The general form of the compactly supported radial basis functions are defined as follows:

$$\phi_{l,k}(r)=(1-r)_+^n p(r),$$