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An efficient adaptive analysis procedure using the edge-based smoothed point interpolation method (ES-PIM) for 2D and 3D problems

Q. Tang [a,b], G.Y. Zhang [c,*], G.R. Liu [d], Z.H. Zhong [a], Z.C. He [a,b]

[a] State Key Laboratory of Advanced Design and Manufacturing for Vehicle Body, Hunan University, Changsha 410082, China
[b] Department of Mechanical Engineering, National University of Singapore, 9 Engineering Drive 1, 117576 Singapore, Singapore
[c] School of Mechanical and Chemical Engineering, The University of Western Australia, 35 Stirling Highway, WA 6009, Australia
[d] School of Aerospace Systems, University of Cincinnati, Cincinnati, OH 45221-0070, USA

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A B S T R A C T

In this paper, an efficient adaptive analysis procedure is proposed using the newly developed edge-based smoothed point interpolation method (ES-PIM) for both two dimensional (2D) and three dimensional (3D) elasticity problems. The ES-PIM works well with three-node triangular and four-node tetrahedral meshes, is easy to be implemented for complicated geometry, and can obtain numerical results of much better accuracy and higher convergence rate than the standard finite element method (FEM) with the same set of meshes. All these important features make it an ideal candidate for adaptive analysis. In the present adaptive procedure, a novel error indicator is devised for ES-PIM settings, which evaluates the maximum difference of strain energy values among the vertexes of each background cell. A simple *h*-type local refinement scheme is adopted together with a mesh generator based on Delaunay technology. Intensive numerical studies of 2D and 3D examples indicate that the proposed adaptive procedure can effectively capture the stress concentration and solution singularities, carry out local refinement automatically, and hence achieve much higher convergence for the solutions in strain energy norm compared to the general uniform refinement.

1. Introduction

Adaptive analysis is important in the computational methods, to achieve desired high accuracy with minimum computational cost. In the traditional FEM, adaptive mesh refinement techniques along with proper error analysis have been well studied [1–5], however for meshfree methods [6–9], it is still an open topic.

The meshfree edge-based point interpolation method (ES-PIM) [10] has been recently developed using the generalized smoothed Galerkin (GS-Galerkin) weak form [11] with the point interpolation method (PIM) for field variable approximation [12] and the edge-based gradient smoothing operation for strain construction. In the ES-PIM, PIM shape functions are constructed with a set of small number of nodes located in a local support domain and possess Kronecker Delta function property which allows straightforward imposition of point essential boundary conditions. The generalized gradient smoothing technique [11] extended form the strain smoothing operation [13] allows the use of discontinuous functions. It can provide the so-called "softening" effect to the numerical model and hence solve the overly-stiff problem existing in a displacement-based fully compatible FEM model [11,14]. As the theoretical basis of ES-PIM, Liu and coworkers have developed the G space theory and the weakened weak (W^2) formulation [11,15,16] for a unified formulation of a wide class of compatible and incompatible methods. These methods include the present ES-PIM, the node-based smoothed point interpolation method (NS-PIM or LC-PIM originally) which can provide upper bound solutions in energy norm for the force driven elasticity problems [14,17,18], the cell-based smoothed point interpolation method (CS-PIM) which obtains highly accurate and convergent solutions [16,46] and strain constructed point interpolation method (SC-PIM) [47,48]. For the ES-PIM, or the edge-based smoothed finite element method (ES-FEM) which is a special case of ES-PIM using linear shape functions [19], the edge-based strain smoothing operation can properly soften the model and make the numerical model have a quite close-to-exact stiffness by even using linear triangular elements [10,19]. Thus numerical solutions by the ES-PIM are generally of much better accuracy, higher convergence rate and efficiency than the standard FEM using the same mesh. Furthermore, the formulation of the ES-PIM is straightforward, the implementation is very simple and the method works well particularly for low-order linear elements. All these features make the ES-PIM an excellent candidate for adaptive analyses.

* Corresponding author.
E-mail address: guiyong@gmail.com (G.Y. Zhang).

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In an adaptive analysis an appropriate error indicator and associated mesh refinement strategy are two crucial issues. In general, two distinct types of procedures are currently available for deriving error indicators: the recovery based error indicator and the residual based error indicator. Zienkiewicz and Zhu [5] firstly introduced the well-known recovery based error indicator in 1987, which uses recovered stresses as reference solutions to calculate the domain error in energy norm. There are some excellent studies on this recovery methodology [20–22]. Residual based error indicators make use of the residual of the numerical approximation, either explicitly or implicitly, which offers a very effective alternative [23–25].

A local mesh refinement is according to estimated error distributions. The basic adaptive refinement schemes can be categorized into *h*-refinement, *p*-refinement and *r*-refinement [26]. The *h*-refinement scheme changes the size of element in a localized fashion based on the error indicator. The *p*-refinement scheme is to increase the order of the polynomial, and the *r*-refinement keeps the total number of nodes unchanged but to adjust their positions to obtain an optimal approximation. Considering the fact that the ES-PIM works particularly well with low-order interpolation of displacement field, and the simplicity and effectiveness of *h*-refinement, we use the *h*-refinement scheme in the present adaptive analysis.

In the scheme of meshfree method, a number of adaptive analysis procedures and error analysis techniques have been proposed. For the element-free Galerkin (EFG) method, Rabczuk and Belytschko [27] proposed adaptive analysis for structured meshfree particle methods in 2D and 3D problems. Duarte and Oden [28] derived an error indicator that involves the computation of the interior residuals and the residuals for Neumann boundary conditions for hp-cloud method. Liu and Tu [29] have introduced an adaptive procedure for meshfree methods using an error indicator of energy error computed via different order of sampling in Gauss integration based on background cells. Angulo et al. [30] implemented adaptive produce of meshfree finite point method for solving boundary value problems. An efficient adaptive RKPM for 3D contact problems with elastic–plastic dynamic large deformation was proposed by Gan et al. [31]. Zhang et al. [32] have conducted adaptive analysis with the node-based smoothed point interpolation method (NS-PIM or LC-PIM originally) to certify solutions with exact bounds of strain energy for 2D problems. Then Tang et al. [33] have extended the NS-PIM adaptive procedure to 3D problems.

In this paper, we propose an efficient error indicator and the associated refinement scheme within the framework of the ES-PIM and FEM for adaptive analysis for both 2D and 3D problems. The proposed error indicator is defined based on the maximum differences of strain energy among the vertexes associated with each background cell. A simple *h*-type refinement scheme is then implemented with an effective strategy for adding in nodes into the regions identified by the error indicator. The automatic 2D and 3D mesh generators based on Delaunay technology are next coded to regenerate meshes for each step in the adaptive process. Adaptive analysis is finally performed for a number of 2D and 3D problems, including ones with stress concentration and singularities. The results demonstrate that the present adaptive procedure performs very well for the ES-PIM to obtain solutions of desired accuracy and with bounds to the exact solution.

The layout of this paper is as follows. In Section 2, the basic equations of ES-PIM are given. Section 3 describes the proposed adaptive procedure, including the definition of error indicator, the calculation of local critical value, the strategy of *h*-type refinement and the automatic Delaunay mesh generation. In Section 4, some numerical problems in 2D and 3D are conducted to assess the capabilities of the proposed adaptive procedure. Conclusions are stated in Section 5.

2. Briefing on the ES-PIM

2.1. Basic equations

Consider a solid mechanics problem in domain Ω bounded by $\Gamma(\Gamma=\Gamma_t+\Gamma_u)$. The standard strong form governing equations can be expressed by the following equations [34]:

Equilibrium equation:

$$\mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma}+\mathbf{b}=0 \text{ in } \Omega \tag{1}$$

where $\mathbf{L}$ is a differential operator in the following forms:

$$\mathbf{L}^{\mathrm{T}}=\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \text{ (for 2D problems)}$$

$$\mathbf{L}^{\mathrm{T}}=\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \text{ (for 3D problems)} \tag{2}$$

$\boldsymbol{\sigma}^{\mathrm{T}}=\left\{ \sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy} \right\}$ for 2D problems and $\boldsymbol{\sigma}^{\mathrm{T}}=\left\{ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx} \right\}$ for 3D problems are the vectors containing stress components respectively, and $\mathbf{b}^{\mathrm{T}}=\left\{ b_x \quad b_y \right\}$ and $\mathbf{b}^{\mathrm{T}}=\left\{ b_x \quad b_y \quad b_z \right\}$ are the external body force vectors for 2D and 3D problems respectively.

Essential boundary conditions:

$$\mathbf{u}=\mathbf{u}_p \text{ on } \Gamma_u \tag{3}$$

where $\mathbf{u}^{\mathrm{T}}=\{ u \quad v \}$ and $\mathbf{u}^{\mathrm{T}}=\{ u \quad v \quad w \}$ are the displacement vectors for 2D and 3D problems respectively, and $\mathbf{u}_p$ is the prescribed displacements on the essential boundaries.

Natural boundary conditions:

$$\boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}_p \text{ on } \Gamma_t \tag{4}$$

where $\mathbf{t}_p$ is the prescribed traction on the natural boundaries, and $\mathbf{n}$ is the vector of unit outward normal on Γ_t.

2.2. Construction of PIM shape functions

PIM shape functions are constructed using a set of small number of nodes located in a local support domain [8]. There are two types of PIM shape functions which have been developed with different basis functions, *i.e.* polynomial basis functions [8,12] and radial basis functions [8,35]. Details of construction PIM shape functions can be found in the book by Liu [8]. In this work we use both the simplest linear polynomial basis functions and radial basis functions to construct PIM shape functions.

For the polynomial PIM, the formulations start with the following assumption:

$$u(\mathbf{x})=\sum_{i=1}^{n} P_i(\mathbf{x})a_i=\mathbf{P}^T(\mathbf{x})\mathbf{a}, \tag{5}$$

where $u(\mathbf{x})$ is a field variable function defined in the Cartesian coordinate space, $P_i(\mathbf{x})$ is the basis function of monomials which is usually built utilizing Pascal's triangles, a_i is the corresponding coefficient, and n is the number of nodes in the local support domain.

For the radial PIM, using radial basis functions augmented with polynomials, a field variable function $u(\mathbf{x})$ can be approximated as follows:

$$u(\mathbf{x})=\sum_{i=1}^{n} R_i(\mathbf{x})a_i+\sum_{j=1}^{m} P_j(\mathbf{x})b_j=\mathbf{R}^T(\mathbf{x})\mathbf{a}+\mathbf{P}^T(\mathbf{x})\mathbf{b}, \tag{6}$$