Transportation Research Part B 94 (2016) 204–216

Contents lists available at ScienceDirect

Transportation Research Part B

journal homepage: www.elsevier.com/locate/trb

# The impact of source terms in the variational representation of traffic flow

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ARTICLE INFO

*Article history:*
Available online 5 October 2016

*Keywords:*
Traffic flow
Source terms
Kinematic wave model

ABSTRACT

This paper revisits the variational theory of traffic flow, now under the presence of continuum lateral inflows and outflows to the freeway. It is found that a VT solution apply only in Eulerian coordinates when source terms are exogenous, but not when they are a function of traffic conditions, e.g. as per a merge model. In discrete time, however, these dependencies become exogenous, which allowed us to propose improved numerical solution methods. In space-Lagrangian and time-Lagrangian coordinates, VT solutions may not apply even if source terms are exogenous.



## 1. Introduction

The variational theory (VT) of traffic flow theory (Claudel and Bayen, 2010a; 2010b; Daganzo, 2005a; 2005b) was an important milestone. Previously, the only analytical solution to the kinematic wave model of Lighthill and Whitham (1955); Richards (1956) was obtained thanks to the method of characteristics, which does not give a global solution in time as one needs to keep track of characteristics crossings (shocks and rarefaction waves) and impose entropy conditions to ensure uniqueness. This means that analytical solutions cannot be formulated except for very simple problems.

In contrast, VT makes use of the link between conservation laws and the Hamilton–Jacobi partial differential equation (HJ PDE), allowing the kinematic wave model to be solved using the Hopf–Lax formula (Hopf, 1970; Lax, 1957; Olejnik, 1957), better known in transportation as Newell's minimum principle (Newell, 1993) when the fundamental diagram is piecewise triangular. The big advantage is that this representation formula gives an analytical global solution in time that does not require explicit consideration of shocks and/or entropy conditions. Moreover, current approximation methods for the Macroscopic Fundamental Diagram (MFD) of urban networks rely on this approach (Daganzo and Geroliminis, 2008; Geroliminis and Boyacı, 2013; Laval and Castrillón, 2015; Leclercq and Geroliminis, 2013). In addition, when the flow-density fundamental diagram is triangular (or piecewise linear more generally), numerical solutions become exact in the HJ framework, and this is not the case in the conservation law approach (LeVeque, 1993). It has also been shown that the traffic flow problem cast in Lagrangian or vehicle number-space coordinates also accepts a VT solution, which can be used to obtain very efficient numerical solution methods (Laval and Leclercq, 2013; Leclercq et al., 2007).

The traffic flow problem with source term is also of great interest; e.g., it can be used to approximate (i) long freeways with closely spaced entrances and exits, (ii) the effect of lane-changing activity on a single lane, or (iii) the effects of turning movements, trip generation and trip ends in the MFD. But the underlying assumption in the previous paragraphs is that

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http://dx.doi.org/10.1016/j.trb.2016.09.011

vehicles are conserved. This begs the question, are representation formulas for the VT solutions still valid, or even applicable, when there is a Eulerian source term? If not, can efficient numerical solution methods still be implemented? Recent developments in this area have not answered these questions as they are primarily concerned with discrete source terms (e.g., Daganzo, 2014; Costeseque and Lebacque, 2014a; 2014b; Li and Zhang, 2013).

To answer these questions this paper is organized as follows. In Section 2 we formulate the general problem and show that in general VT solutions are not applicable; but Section 3 shows that they are when the source term is exogenous. Based on these results, Section 4 presents numerical methods for the endogenous inflow problem that outperform existing methods. Section 5 briefly shows that in space-Lagrangian and time-Lagrangian coordinates VT solutions do not exist even if source terms are exogenous. A discussion of results and outlook is presented in Section 6.

## 2. Problem formulation

Consider a long homogeneous freeway corridor with a large number of entrances and exits such that the net lateral freeway inflow rate, $\phi$, or inflow for short, can be treated as a continuum variable in time $t \geq 0$ and location $x \geq 0$, and has units of veh/time-distance. The inflow is an endogenous variable consequence of the demand for travel, and could be captured by a function of the traffic states both in the freeway and the ramps. For simplicity, in this paper the inflow is assumed to be a function of the density, $k(t, x)$, of the freeway only, i.e. $\phi = \phi(k(t,x))$, but also the exogenous case $\phi = \phi(t,x)$ will be of interest.

In any case, the traffic flow problem analyzed in this paper is the following conservation law with source term:

$$k_t + H(k)_x = \phi, \tag{1a}$$

$$k = g \quad \text{on} \quad \Gamma \tag{1b}$$

where $H$ is the fundamental diagram, $g$ is the data defined on a boundary $\Gamma$, and variables in subscript represent partial derivatives. Now, let the function $N(t, x)$ be defined as:

$$N(t,x) = \int_x^{+\infty} k(t,y)dy, \tag{2}$$

and note that we have $N_x = -k$ as usual. Integrate (1) with respect to $x$ to obtain its HJ form (Evans, 1998):

$$N_t - H(-N_x) = \Phi, \tag{3a}$$

$$N = G \quad \text{on} \quad \Gamma, \tag{3b}$$

where we have defined:

$$G(t,x) = \oint_\Gamma g(t,x)d\Gamma, \qquad (t,x) \in \Gamma, \qquad \text{and} \tag{4a}$$

$$\Phi(t,x) = -\int_0^x \phi(t,y)dy \tag{4b}$$

where $\Phi$ is a potential function; the negative sign in (4b) follows from the traditional counting convention in traffic flow, where further downstream vehicles have lower vehicle numbers.

It is important to note that compared to the traditional case with zero inflow, under a continuum source term the interpretation of $N$ changes: for fixed $x$ it is no longer a cumulative count curve and its isometrics no longer give vehicle trajectories. To see this we note that according to (3a) the flow $q = H(k)$ is now:

$$q = N_t - \Phi, \tag{5}$$

The time integration of (5) reveals that cumulative count curves are now given, up to an arbitrary constant, by $N(t,x) + \int_0^t \int_0^x \phi(s,y)dsdy$, and the integral represents the net number of vehicles entering the road segment by time $t$ and upstream of $x$.

The method to obtain the solution of (3) depends on the dependencies of the potential function. If the inflow function is allowed to depend on the traffic state in the freeway, i.e. $\phi = \phi(k)$ then the potential function $\Phi$ depends non-locally on $N$, since:

$$\Phi(t,x) = \tilde{\Phi}(N,x) = -\int_0^x \phi(-N_x(t,y))dy, \tag{6}$$

and therefore it is not obvious that (3) accepts a VT solution. To see this, we note that even in the simplest linear case:

$$\phi(k) = a - bk, \qquad a,b \geq 0, \qquad \text{we get:} \tag{7a}$$