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A new look at the rate of change of energy consumption with respect to journey time on an optimal train journey

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ABSTRACT

We present a new derivation of a key formula for the rate of change of energy consumption with respect to journey time on an optimal train journey. We use a standard mathematical model (Albrecht et al., 2015b; Howlett, 2000; Howlett et al., 2009; Khmelnitsky, 2000; Liu and Golovitcher, 2003) to define the problem and show by explicit calculation of switching points that the formula also applies for all basic control subsequences within the optimal strategy on appropriately chosen fixed track segments. The rate of change was initially derived as a known strictly decreasing function of the optimal driving speed in a text edited by Isayev (1987, Section 14.2, pp 259–260) using an empirical resistance function. An elegant derivation by Liu and Golovitcher (2003, Section 3) with a general resistance function required an underlying assumption that the optimal strategy is unique and that the associated optimal driving speed is a strictly decreasing and continuous function of journey time. An earlier proof of uniqueness (Khmelnitsky, 2000) showed that the optimal driving speed decreases when journey time increases. A subsequent constructive proof (Albrecht et al., 2013a, 2015c) used a local energy minimization principle to find optimal switching points and show explicitly that the optimal driving speed is a strictly decreasing and continuous function of journey time. Our new derivation of the key formula also uses the local energy minimization principle and depends on the following observations. If no speed limits are imposed the optimal strategy consists of a finite sequence of phases with only five permissible control modes. By considering all basic control subsequences and subdividing the track into suitably chosen fixed segments we show that the key formula is valid on each individual segment. The formula is extended to the entire journey by summation. The veracity of the formula is demonstrated with an elementary but realistic example.

1. Introduction

We consider an optimal driving strategy for a train journey between two fixed locations on a track with continuous gradient. For each feasible journey time the optimal driving strategy is the one which minimizes the mechanical energy required to drive the train. If we assume that no artificial speed limits are imposed then on track with steep gradients the optimal strategy is a complex switching strategy using only five permissible optimal control modes—*Maximum Acceleration*, *Speedhold with Partial Acceleration at the Optimal Driving Speed*, *Coast*, *Speedhold with Partial Brake at the Optimal Regenerative*

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Braking Speed and *Maximum Brake*. The optimal strategy is completely determined by the optimal driving speed. See Albrecht et al. (2015b, 2015c) for a comprehensive discussion. If artificial speed limits are imposed then the optimal strategy may include additional segments of singular control where the train follows the imposed speed limit (Albrecht et al., 2015b; 2015c; Khmelnitsky, 2000; Liu and Golovitcher, 2003). Although the structure of the optimal driving strategy for a single train is now well understood (Albrecht et al., 2013a; 2015b; 2015c; Howlett, 2000; Howlett et al., 2009; Khmelnitsky, 2000; Liu and Golovitcher, 2003) the detailed computational techniques suggested in these seminal papers and now used for on-board calculation of optimal strategies in real time[1] are apparently not widely known. Indeed many papers still argue that real time calculations are only possible if one assumes grossly simplified driving strategies.

In this paper we use the known structure of an optimal strategy on steep track with no speed limits to present a new derivation of a key formula for the rate of change in energy consumption with respect to journey time. The main argument is outlined in the body of the paper but the detailed derivation requires complicated mathematical manipulation of known formulæ directly related to the local energy minimization principle (Albrecht et al., 2013a; 2015b; 2015c; Howlett et al., 2009). For the convenience of readers these mathematical complications have been deferred to the appendices. We refer readers to Isayev (1987, Section 14.2, pp 259–262) and Liu and Golovitcher (2003, Section 3) for earlier derivations of the key formula.

1.1. Main contribution

For convenience we shall refer to the graph of mechanical energy consumption against journey time for the optimal driving strategy as the cost-time curve. Our main contribution is to present a new derivation of the key formula

$$\frac{dJ}{dT} = -\psi(V) < 0 \iff J'(V) = -\psi(V)T'(V) > 0 \tag{1}$$

where $V > 0$ is the optimal driving speed, $J = J(V)$ is the mechanical energy consumption or cost and $T = T(V)$ is the time taken for the journey and to show by explicit calculation of the switching points that this relationship is preserved over all basic permissible control subsequences on suitably chosen fixed segments of the optimal journey. The function $\psi(v) = v^2 r'(v)$ is a non-negative strictly increasing function that depends only on the resistive acceleration per unit mass $r(v)$ at speed v. The formula (1) shows that the *so-called* cost-time curve is *strictly monotone decreasing* and *strictly convex*. The optimal driving speed V uniquely determines the optimal driving strategy and although it is nominally associated with an optimal *speedhold* segment it is important to understand that on steep tracks or on short journeys there are many instances where the optimal strategy contains no *speedhold* segment. Furthermore it is well known that on tracks with complex gradient profiles the optimal driving strategy may involve an equally complex sequence of optimal controls and associated optimal switching points where the control changes.

The simplicity of the key formula (1) may seem somewhat surprising when one considers the potential complexity of the optimal driving strategy. However the explanation in Liu and Golovitcher (2003, Section 3) shows that the formula follows naturally from the necessary conditions for optimality provided that the optimal strategy is uniquely defined and that the associated optimal driving speed is a continuous and strictly decreasing function of the journey time. In this regard the main virtue of our new derivation is that it relates directly to the determination of optimal switching points using the local energy minimization principle and hence also to the constructive proof by Albrecht et al. (2013a); 2015c) that the optimal strategy is unique.

1.2. Motivation and a related result

Our study is motivated in the first instance by a typical train journey in which a train is required to stop at a succession of intermediate stations while travelling from an initial station to a final station. It is common practice to seek a driving strategy that minimizes the mechanical energy required to drive the train between consecutive stops subject to a prescribed section running time. In order to minimize energy consumption for an entire journey subject to an overall allowed journey time the section running times must also be chosen appropriately (Isayev, 1987, Section 14.2, pp 260–262). Hence it is useful to understand that the structure of the *so-called* cost-time curves on each timed journey section can be used to derive the following auxiliary result.

Let $J_i = J_i(V_i)$ denote the cost and $T_i = T_i(V_i)$ denote the duration of the optimal journey on section S_i where V_i is the optimal driving speed for each $i = 1, \ldots, n$. Write $V = (V_1, \ldots, V_n)$. We wish to minimize the total journey cost

$$J(V) = J_1(V_1) + \cdots + J_n(V_n)$$

subject to $T_1(V_1) + \cdots + T_n(V_n) \le T$. If we set

$$\mathcal{J}(V) = J(V) + \lambda[T_1(V_1) + \cdots + T_n(V_n) - T]$$

then the Karush–Kuhn–Tucker (KKT) conditions show that

$$J_i'(V_i) = -\lambda T_i'(V_i) \tag{2}$$

[1] See, for instance, the *Energymiser* technology at www.ttgtransportationtechnology.com.