Engineering Analysis with Boundary Elements 36 (2012) 394–403

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

The method of fundamental solutions and its combinations for exterior problems ☆

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ARTICLE INFO

Article history:
Received 15 December 2010
Accepted 28 July 2011
Available online 26 October 2011

Keywords:
Method of fundamental solutions
Method of particular solutions
Fundamental solutions
Trefftz method
Exterior problem
Singularity problem

ABSTRACT

Most of the reports on the method of fundamental solutions (MFS) deal with bounded simply connected domains; only a few involve exterior problems. For exterior problems governed by Laplace's equation, there exist two kinds of infinity conditions, (1) $|u| < C$ and (2) $u = O(\ln \rho)$. For $u = O(\ln \rho)$, the traditional fundamental solutions can be used. However, for $|u| \le C$, new fundamental solutions are explored. Numerical experiments are carried out to verify the theoretical analysis. The MFS and the method of particular solutions (MPS) are classified as Trefftz methods (TM) [30] using fundamental solutions (FS) and particular solutions (PS), respectively. The remarkable advantage of the MFS over the MPS is the uniform FS: $\ln r = \ln|\overline{PQ}|$, where P and Q are the solution and the source points, respectively. Hence both algorithms and programming are simple. Moreover, a crack singularity in unbounded domains (i.e., exterior problem) is also studied. A combination of the TM using both PS and FS is also employed. The numerical results of the MPS and the combination of MFS and MPS coincide with each other. The study in this paper may greatly extend the application of the MFS from bounded simply connected domains to exterior domains.

1. Introduction

The method of fundamental solutions (MFS) was first used in Kupradze [16] in 1963. Since then, there have appeared numerous reports on MFS for computation, see the reviews of the MFS in Fairweather and Karageorghis [9], Golberg and Chen [11], and a systemic introduction on the MFS in Chen et al. [5]. The MFS has been applied to Cauchy and Stokes problems in [31,33], the biharmonic equation in [8,9], and even to a nonlinear Poisson problem [1]. Some important properties of the MFS were addressed by Schaback [29]. To celebrate the progress of the MFS, the first Workshop on the Method of Fundamental Solutions (MFS2007) (see [6]), was held in Ayia Napa, Cyprus, June 11–13, 2007, and the second Workshop joined with Workshop on Trefftz IV, was held in Kaoksiung, Taiwan, March 15–18, 2011. On the other hand, the Trefftz method (TM) [30] has been fully developed in theory and computation for several decades (see [25]), where only the particular solutions (PS) are used. In fact, the MFS is a TM using fundamental solutions (FS). In order to distinguish the two methods, in this paper, the TM using PS is called the method of particular solutions (MPS), as in Betcke and Trefethen [3].[1] Such a method is also called the particular solution Trefftz method in [2]. Both the MFS and the MPS belong as TM [25].

Most of the reports on the MFS deal with bounded simply connected domains; only a few papers involve exterior problems (e.g., [5,6,15,28,14,10,31]). For exterior problems governed by Laplace's equation, there exist two kinds of infinity conditions, (1) $|u| < C$ and (2) $u = O(\ln \rho)$. Suitable fundamental solutions must be found. For $u = O(\ln \rho)$, the traditional fundamental solutions can be used. However, for $|u| \le C$, new fundamental solutions should be explored. This is the first goal of this paper. Since the method of fundamental solutions (MFS) can be classified as a Trefftz method (TM) using fundamental solutions (FS). Then we may follow [25] to obtain the error analysis of the MFS. The remarkable advantage of the MFS over the MPS is the uniform FS: $\ln r = \ln|\overline{PQ}|$, where P and Q are the solution and the source points, respectively. Hence both algorithms and programming are simple. The MFS may satisfy the engineering requirements by much less computational efforts. The second goal is to study a challenging model: a crack singularity problem in an unbounded

☆ Partial results of this paper were presented at Progress in Electromagnetics Research Symposium (PIERS) 2009, Moscow, Russia, 18–21 August 2009.

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[1] In Chen et al. [5], the method of particular solutions is referred only to the method to seek the special particular solutions of nonhomogeneous equations, such as Poisson's equation.

0955-7997/$ - see front matter © 2011 Published by Elsevier Ltd.
doi:10.1016/j.enganabound.2011.07.016

domain (i.e., exterior problem) is studied. A combination of the PS and FS is also employed. The numerical results of the MPS and the combination of MFS and MPS are coincident with each other. The study in this paper may greatly facilitate the application of the MFS from bounded simply connected domains to exterior domains.

In this paper, we focus on unbounded domains, and discuss both smooth and singularity problems. In Section 2, the Trefftz method is described for Laplace's equation. When the fundamental solutions and the particular solutions are used in the TM, we obtain the MFS and the MPS, respectively. In Section 3, for the exterior problem with smooth solutions, new fundamental solutions satisfying the infinity condition $|u|<C$ as $\rho\to\infty$ are explored. In Section 4, the crack singularity problem in unbounded domains is solved by the MPS and the combination of MFS and MPS, where the infinity condition is $u=O(\ln\rho)$ as $\rho\to\infty$.

2. The Trefftz method

Consider the exterior problem in 2D,

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{on } S_\infty, \tag{2.1}$$

$$u=f \quad \text{on } \Gamma_{in}^D, \tag{2.2}$$

$$u_\nu=\frac{\partial u}{\partial \nu}=g \quad \text{in } \Gamma_{in}^N, \tag{2.3}$$

$$|u|\le C \text{ or } u=O(\ln\rho) \quad \text{as } \rho\to\infty, \tag{2.4}$$

where S_∞ is the exterior domain with the interior boundary

$$\Gamma_{in}=\Gamma_{in}^D\cup\Gamma_{in}^N, \tag{2.5}$$

ν is the exterior normal to Γ_{in}^N, and f and g are smooth functions. The infinity condition as $u=O(\ln\rho)$ for $\rho\to\infty$ is given in McLean [28], resulting from some boundary integral equations, see also Section 4.

We choose the harmonic functions ϕ_i (e.g., $\Delta\phi_i=0$) as basis functions, and form a linear combination,

$$u_N=\sum_{i=1}^N c_i\phi_i, \tag{2.6}$$

where c_i are the coefficients to be determined by (2.2)– (2.4). Note that the admissible functions u_N chosen in (2.6) must satisfy the infinity conditions. We may invoke the Trefftz method (TM), which reads: To seek u_N such that

$$I(u_N)=\min_{v\in V_N} I(v), \tag{2.7}$$

where V_N denotes the set of (2.6) satisfying (2.4), the integrals

$$I(v)=\int_{\Gamma_{in}^D}(v-f)^2+w^2\int_{\Gamma_{in}^N}(v_\nu-g)^2, \tag{2.8}$$

and w is the weight. In computations, we may choose

$$w=\frac{1}{N}, \tag{2.9}$$

based on the analysis in [18]. Eq. (2.8) reflects (2.2) and (2.3) only; the condition (2.4) at infinity confines that the functions u_N are bounded in S_∞, or $O(\ln\rho)$ as $\rho\to\infty$.

When the integrals in (2.8) involve numerical approximation, the TM reads: To seek $\tilde{u}_N$ such that

$$\widehat{I}(\tilde{u}_N)=\min_{v\in V_N}\widehat{I}(v), \tag{2.10}$$

where

$$\widehat{I}(v)=\widehat{\int}_{\Gamma_{in}^D}(v-f)^2+w^2\widehat{\int}_{\Gamma_{in}^N}(v_\nu-g)^2, \tag{2.11}$$

and $\widehat{\int}_{\Gamma_{in}^D}$ and $\widehat{\int}_{\Gamma_{in}^N}$ of $\int_{\Gamma_{in}^D}$ and $\int_{\Gamma_{in}^N}$ are evaluated numerically, respectively. On the other hand, we may formulate the collocation equations, directly from (2.2) and (2.3):

$$\sum_{i=1}^N c_i\phi_i(P_j)=f(P_j), \quad P_j\in\Gamma_{in}^D, \tag{2.12}$$

$$\sum_{i=1}^N c_i\frac{\partial}{\partial\nu}\phi_i(P_j)=g(P_j), \quad P_j\in\Gamma_{in}^N. \tag{2.13}$$

Let Γ_{in}^D and Γ_{in}^N be divided into small sections with a meshspacing Δh_j, and denote their mid-points by P_j. We rewrite (2.12) and (2.13) with the weights $\sqrt{\Delta h_j}$ and $w\sqrt{\Delta h_j}$, to give

$$\sqrt{\Delta h_j}\sum_{i=1}^N c_i\phi_i(P_j)=\sqrt{\Delta h_j}f(P_j), \quad P_j\in\Gamma_{in}^D, \tag{2.14}$$

$$w\sqrt{\Delta h_j}\sum_{i=1}^N c_i\frac{\partial}{\partial\nu}\phi_i(P_j)=w\sqrt{\Delta h_j}g(P_j), \quad P_j\in\Gamma_{in}^N. \tag{2.15}$$

Eqs. (2.14) and (2.15) are just (2.10) and (2.11), evaluated by the central rule. We may also obtain the collocation equations by the Gaussian rule. Note that when the fundamental solutions (FS) and the particular solutions (PS) are chosen as the harmonic functions ϕ_i in (2.6), the method of fundamental solutions (MFS) and the method of particular solutions (MPS) are obtained, respectively.

In computation, we always choose the number M of collocation nodes larger than N. Eqs. (2.14) and (2.15) form an over-determined system,

$$\mathbf{Fx}=\mathbf{b}, \tag{2.16}$$

where $\mathbf{F}\in R^{M\times N}(M\ge N)$, $\mathbf{x}\in R^N$ and $\mathbf{b}\in R^M$. The traditional condition number is given by

$$\text{Cond}=\frac{\sigma_{\max}}{\sigma_{\min}}, \tag{2.17}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximal and the minimal singular values of the matrix $\mathbf{F}$, respectively. The effective condition number is defined in [21,22]

$$\text{Cond\_eff}=\frac{\|\mathbf{b}\|}{\sigma_{\min}\|\mathbf{x}\|}, \tag{2.18}$$

where $\|\mathbf{x}\|$ is the Euclidean norm. The values of Cond_eff are smaller (or even much smaller) than those of Cond (see [21,22]).

In (2.16), we choose the over-determined system of linear algebraic equations, to earn a flexibility of the TM, and to reach a better accuracy of the leading coefficients by the MPS. For Motz's problem, the most accurate leading coefficients within rounding errors can be achieved by some over-determined system, see [25].

3. Smooth model

We choose the epitrochoid boundary shape Γ_{in} from Liu [27],

$$\rho(\theta)=\sqrt{(a+b)^2+1-2(a+b)\cos\left(\frac{a\theta}{b}\right)}, \tag{3.1}$$

where $x=\rho\cos\theta$ and $y=\rho\sin\theta$. In our computation, choose $a=3$ and $b=1$ (see Fig. 1). Denote

$$r_{\max}=\max_{\Gamma_{in}}\rho=5, \quad r_{\min}=\min_{\Gamma_{in}}\rho=3. \tag{3.2}$$