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Boundary element analysis of Navier–Stokes equations

J. Matsunashi [a,1], N. Okamoto [b,2], T. Futagami [c,*,2]

[a] Faculty of Engineering, Kobe University, Japan
[b] Department of Applied Chemistry and Biotechnology, Faculty of Engineering, Okayama University of Science, Japan
[c] Department of Civil Engineering and Urban Design, Faculty of Engineering, Hiroshima Institute of Technology, Miyake, Saeki-cho, Hiroshima-city 731-5193, Japan

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ABSTRACT

As arrangements, the fundamental solutions of anisotropic convective diffusion equations of transient incompressible viscous fluid flow and boundary elements analysis of the diffusion equation are presented. Secondly, by considering that convective diffusion equations and Navier–Stokes equations are mathematical formulations of mass and momentum conservation law respectively, and that consequently, both physical contents and equation styles are analogous, boundary integral formulations for Navier–Stokes equations are proposed on the basis of formulation of diffusion equations.

1. Introduction

In general, a Green's function or principal solution (fundamental solution) plays an important role in the conversion of equations into integral equations. The Navier–Stokes equation (hereinafter, abbreviated as N–S equation) is a non-linear equation. Conventional fundamental solutions are structured in such a way that they are intended only for linear operations of equations. Accordingly, the equation is handled in a manner similar to the Stokes approximation, which is known as a linear approximation of the N–S equation, or the Oseen approximation, with many researches utilizing this technique [1–3]. These linearizations are useful for cases where the Reynolds number is extremely small, but their application for general flow in rivers and oceans may be considered to be limited.

This research consists of three steps. In Step 1, velocity components, which are coefficients comprising three-dimensional transient anisotropic convective diffusion equations, are considered to be temporarily given. As a result, fundamental solutions for linear equations are derived. In Step 2, this fundamental solution is used to establish the boundary element method of the convective diffusion equation. It is a well-known fact that convective diffusion equations are mathematical formulae of the law of conservation of mass, while N–S equations are formulae of the law of conservation of momentum, with analogous physical contents and style of equation. An observation is made on the aforementioned characteristics in Step 3 to establish the solving method of BEM for the N–S equation based on the results obtained in Steps 1 and 2.

2. Fundamental solution for convective diffusion equations (Step 1)

A three-dimensional transient anisotropic convective diffusion equation is provided by the following equation:

$$\frac{\partial c}{\partial t}+u_i c,_i = D_i c,_{ii} - \lambda c + F \tag{1}$$

A tensor syntax and sum rule are adopted with a sum of 1, 2 and 3 represented by suffixes in cases where there are multiple identical lower suffixes. Variable t represents time and c represents concentration, while u_1,u_2 and u_3 represent components of flow velocity and D_1,D_2 and D_3 represent the diffusion coefficients in the x_1,x_2 and x_3 directions respectively. λ represents the dissipation rate of mass, while F represents the load term. Suffix $(),_i$ represents differentiation in the direction i, while $(),_{ii}$ represents multiple differentiation in the direction i.

Operator $L[\,]$ is defined as shown in the following equation:

$$L[c] = \frac{\partial c}{\partial t}+u_i c,_i + \lambda c - D_i c,_{ii} \tag{2}$$

The adjoint differential operator of $L[\,]$ is defined as $L^+[\,]$ and the adjoint unknown quantity of c is defined as c^* thus, we obtain the following equation:

$$L^+[c^*] = -\frac{\partial c^*}{\partial t} - u_i c^*,_i + \lambda c^* - D_i c^*,_{ii} \tag{3}$$

* Corresponding author.
E-mail address: t.futagami.w7@it-hiroshima.ac.jp (T. Futagami).
[1] Emeritus Professor.
[2] Professor.

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Based on this definition, Eq. (1) can be written as

$$L[c] = F \tag{4}$$

Next, the solution (fundamental solution) of the following equation, c^* is searched after

$$L^+[c^*] = \delta(x_1 - b_1)\delta(x_2 - b_2)\delta(x_3 - b_3)\delta(t - \tau) \tag{5}$$

In this equation, δ represents the Dirac delta function, while $x_i (i = 1,2,3)$ and t represent the coordinates of the review point and time, respectively, with $b_i (i = 1,2,3)$ and τ representing the load point and time, respectively. The solutions of Eq. (5) are obtained already under the assumption that $D_1 = D_2 = D_3$, velocity components u_1,u_2 and u_3 are temporarily and spatially constant [4]. In this paper, the solutions of Eq. (5) are obtained under the assumption that $D_1 \neq D_2 \neq D_3$.

$u_j (j = 1,2,3)$ and c^* are both unknown values, and although Eq. (5) is a non-linear partial differential equation, $u_j (j = 1,2,3)$ is temporarily considered to be a known quantity (by providing the flow velocity value for time level t) to linearize the equation (in actual calculation, $u_j (j = 1,2,3)$ is first calculated in Δt steps according to a method described in Step 3, and then, the result is utilized as basic data to implement this diffusion calculation).

The following equation is obtained by performing quadruple Fourier transformation on the above linearized equation with respect to $x_i (i = 1,2,3)$, and t:

$$\hat{c}^*\{-i(\xi_t + u_j\xi_j) + \lambda + D_j\xi_j\xi_j\} = e^{-i\xi_j b_j - i\xi_t\tau} \tag{6}$$

where $i = \sqrt{-1}$ and $\hat{c}^*$ are conversions of unknown quantities, while ξ_1,ξ_2,ξ_3, and ξ_t are conversion variables for x_1,x_2,x_3, and t respectively.

The following equation is obtained by solving the above equation for $\hat{c}^*$:

$$\hat{c}^* = \frac{1}{-i\xi_t + \omega} e^{-i\xi_j b_j - i\xi_t\tau} \tag{7}$$

where

$$\omega = -iu_j\xi_j + \lambda + D_j\xi_j\xi_j \tag{8}$$

Next, by performing quadruple Fourier inverse transformation, we obtain

$$c^* = \frac{1}{(2\pi)^4} \iiint_{-\infty}^{+\infty} e^{i\xi_j r_j}\, d\xi_1\, d\xi_2\, d\xi_3 \int_{-\infty}^{+\infty} \frac{e^{i\xi_t(t-\tau)}}{-i\xi_t + \omega} d\xi_t \tag{9}$$

where $r_j = x_j - b_j$ $(j = 1,2,3)$.

First, the integral for the integration variable ξ_t is searched after

$$I = \int_{-\infty}^{+\infty} \frac{e^{i\xi_t(t-\tau)}}{-i\xi_t + \omega} d\xi_t \tag{10}$$

Next, a simple closed curve on a complex plane is assumed for considering the next complex integral

$$I^* = \oint \frac{e^{izm}}{-iz + \omega} dz \tag{11}$$

where z is a complex number, $m = t - \tau$.

The singular point of I^* is $z = -i\omega$, using which residue A can be written as $A = ie^{\omega m}$. Accordingly, based on the theory of integration we obtain

$$I^* = 2\pi i A = -2\pi e^{\omega m} \tag{12}$$

On the other hand, if the complex plane is defined as $z = \xi + i\eta$, as shown in Fig. 1, then $izm = i\xi m - \eta m$; hence, for the integral I^* to exist, we must have $\eta m > 0$. This occurs in two cases: (1) $m > 0, \eta > 0$; (2) $m < 0, \eta < 0$.

$\overrightarrow{\text{BADB}}$ and $\overrightarrow{\text{ABCA}}$ can be considered as a closed curve that includes real-number axes AB, which are equivalent to cases (1) and (2), respectively.

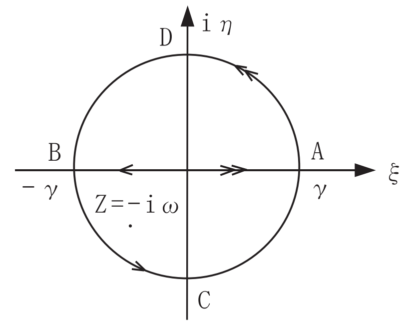

Fig. 1. Integral path on a complex plane.

When considering Eq. (8), the singular point exists in the closed curve ABCA as shown in Fig. 1:

$$z = -i\omega = -u_j\xi_j - i(\lambda + D_j\xi_j\xi_j)$$

Accordingly, the integral values for cases (1) and (2), respectively, are

$$I^* = 0, \quad I^* = -2\pi e^{\omega m}$$

Therefore, to ensure that integral I^* is not 0, we must have $m = t - \tau < 0$. This is merely a natural outcome; because Eq. (5) is fundamentally an expression of diffusion in the reverse direction, i.e., diffusion into the past.

By setting $m < 0$, we obtain

$$I^* = \int_{\text{ABCA}} = \int_{\text{AB}} + \int_{\text{BCA}} = -2\pi e^{\omega m} \tag{13}$$

By setting $\gamma \to \infty$ in Fig. 1 and by applying the theory of Jordan, we obtain

$$\int_{\text{BCA}} = 0, \quad \int_{\text{AB}} = -2\pi e^{\omega m}, \quad I = \int_{\text{BA}} = -\int_{\text{AB}} = 2\pi e^{\omega m} \tag{14}$$

Accordingly, we obtain

$$I = \begin{cases} 2\pi e^{\omega m}, & m < 0 \\ 0, & m > 0 \end{cases} \equiv 2\pi H(-m) e^{\omega m} \tag{15}$$

where $H(-m)$ is a Heaviside step function.

By substituting Eq. (15) in Eq. (9), we obtain

$$c^* = \frac{H(-m)}{(2\pi)^3} \iiint_{-\infty}^{+\infty} e^{i\xi_j r_j} e^{m\omega}\, d\xi_1\, d\xi_2\, d\xi_3 \tag{16}$$

By substituting ω of Eq. (8) in the above equation, we obtain

$$c^* = \frac{e^{m\lambda} H(-m)}{(2\pi)^3} \prod_{j=1}^{3} \int_{-\infty}^{+\infty} e^{i\xi_j r_j - imu_j\xi_j + mD_j\xi_j\xi_j}\, d\xi_j = \frac{e^{m\lambda} H(-m)}{(2\pi)^3} f_1 f_2 f_3 \tag{17}$$

Here, we decide that the sum rule does not apply to the above equation only. f_1, f_2 and f_3 are each calculated separately:

$$f_1 = \int_{-\infty}^{+\infty} e^{i(r_1 - mu_1)\xi_1 + mD_1\xi_1^2}\, d\xi_1 \tag{18}$$

By introducing $\phi^2 = -mD_1 > 0$, $\psi = r_1 - mu_1$, and $x = \xi_1$ to the above equation, we obtain

$$f_1 = \int_{-\infty}^{+\infty} e^{-\phi^2 x^2 + i\psi x}\, dx = \int_{-\infty}^{+\infty} e^{-\phi^2 x^2} \cos\psi x\, dx + i\int_{-\infty}^{+\infty} e^{-\phi^2 x^2} \sin\psi x\, dx \tag{19}$$

The first and second terms on the right side of Eq. (19) are defined as g_1 and g_2, respectively:

$$g_1 = \int_{-\infty}^{+\infty} e^{-\phi^2 x^2} \cos\psi x\, dx = 2\int_0^{+\infty} e^{-\phi^2 x^2} \cos\psi x\, dx$$