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# The null-field method of Dirichlet problems of Laplace's equation on circular domains with circular holes

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ABSTRACT

In this paper, the boundary errors are defined for the null-field method (NFM) to explore the convergence rates, and the condition numbers are derived for simple cases to explore numerical stability. The optimal convergence (or exponential) rates are discovered numerically. This paper is also devoted to seek better choice of locations for the field nodes of the fundamental solutions (FS) expansions. It is found that the location of field nodes $Q$ does not affect much on convergence rates, but do have influence on stability. Let $\delta$ denote the distance of $Q$ to $\partial S$. The larger $\delta$ is chosen, the worse the instability of the NFM occurs. As a result, $\delta = 0$ (i.e., $Q \in \partial S$) is the best for stability. However, when $\delta > 0$, the errors are slightly smaller. Therefore, small $\delta$ is a favorable choice for both high accuracy and good stability. This new discovery enhances the proper application of the NFM.



## 1. Introduction

For circular domains with circular holes, there exist a number of papers of boundary methods. In Babone and Caulk [6,7] and Caulk [10] the Fourier functions are used for the circular holes for boundary integral equations; and in Bird and Steele [8] the simple algorithms as the collocation Trefftz method as in [33] are used. In Ang and Kang [2], complex boundary elements are studied. Recently, Chen and his research group have developed the null-filed method (NFM), in which, the field nodes $Q$ are located outside of the solution domain $S$, and the fundamental solutions (FS) can be expanded as the convergent series. The Fourier functions are also used to approximate the known or unknown Dirichlet and Neumann boundary data, numerous papers have been published for different physical problems. Since explicit algorithms, errors, and stability for Laplace's equation are our main concern, we only cite [15,16,18]. For the boundary integral equation (BIE) of the first kind, the trigonometric functions are used in Arnold [3,4], and error analysis is made for infinite smooth solutions, to derive the exponential convergence rates. In Cheng's Dissertation [21,22], for BIE of the first kind, the field nodes are located outside of the solution domain, the linear combination of fundamental solutions are used, and error analysis is made only for circular domains.

In this paper, the boundary errors are defined for the NFM to explore the convergence rates, and the condition numbers are derived for simple cases to explore numerical stability. The optimal convergence (or exponential) rates are discovered numerically, and a strict error analysis will be reported in a subsequent paper. This paper is also devoted to seek better choice of locations for the field nodes of the FS expansions. In this paper, we apply the indirect BEM with retracted boundary, while in [19], MFS with discrete source nodes are used. It is found that the location of $Q$ does not affect much on convergence rates, but do have influence on stability. Let $\delta$ denote the distance of $Q$ to $\partial S$. The larger the $\delta$ is chosen, the worse the instability of the NFM occurs. As a result, $\delta = 0$ is the best for stability. However, when $\delta > 0$, the errors are slightly smaller. Therefore, small $\delta$ is a better choice for both high accuracy and good stability. This new discovery enhances proper applications of the NFM.

This paper is organized as follows. In the next section, the explicit discrete equations and the analytic solutions of the NFM are derived, and choices of field points in the FS expansions are investigated. In Section 3, stability analysis is made for a simple case of circular boundaries with the same origin. A strict proof is provided for the field nodes located on the solution boundary $\partial S$. In Section 4, numerical experiments are carried out for a model problem of Dirichlet problems of Laplace's equation, and better

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choices of field nodes are found numerically. In the last section, a few concluding remarks are addressed.

## 2. Explicit algorithms of null field methods

### 2.1. Basic algorithms

The series expansions[1] of fundamental solutions (FS) are important to the error analysis of the method of fundamental solutions, and the null-field method (NFM), which is developed by Chen and his colleagues for circular domain with circular holes [15,16,18]. Such special domains can be found in many engineering problems. In order to simply describe the NFM, we confine ourselves Laplace's equation and choose the circular domain with one circular hole. Denote the disks $S_R$ and $S_{R_1}$ with radii $R$ and $R_1$, respectively. Then $S_{R_1} \subset S_R$, and the eccentric circular domains $S_R$ and $S_{R_1}$ may have different origins. Let $2R_1 < R$. The annular solution domain $S = S_R \backslash S_{R_1}$ with the exterior and the interior boundaries $\partial S_R$ and $\partial S_{R_1}$, respectively. In [18], $R$=2.5 and $R_1 = 1$ and the origins of $S_R$ and $S_{R_1}$ are located at (0, 0) and ($-R_1$,0), respectively. The following Dirichlet problems are discussed by Palaniappan [35],

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } S, \tag{2.1}$$

$$u = 1 \ \text{ on } \partial S_R, \quad u = 0 \ \text{ on } \partial S_{R_1}. \tag{2.2}$$

The true solution of (2.1) and (2.2) is found as [18,35]

$$u(\rho,\phi) = \frac{1}{2\ln 2} \ln\left\{\frac{16\rho^2 + 1 + 8\rho\cos\phi}{\rho^2 + 16 + 8\rho\cos\phi}\right\}, \tag{2.3}$$

where $(\rho,\phi)$ are the polar coordinates of $S_{R_1}$ with the origin $(-1,0)$. Eqs. (2.1)–(2.3) are called Model problem in this paper.

On the exterior boundary $\partial S_R$, there exist the approximations of Fourier expansions,

$$u = u_0 := a_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \quad \text{on } \partial S_R, \tag{2.4}$$

$$\frac{\partial u}{\partial \nu} = q_0 := p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \quad \text{on } \partial S_R, \tag{2.5}$$

where $a_k, b_k, p_k$ and $q_k$ are coefficients. On the interior boundary $\partial S_{R_1}$, similarly

$$\overline{u} = \overline{u}_0 := \overline{a}_0 + \sum_{k=1}^{N} \{\overline{a}_k \cos k\overline{\theta} + \overline{b}_k \sin k\overline{\theta}\} \quad \text{on } \partial S_{R_1}, \tag{2.6}$$

$$\frac{\partial \overline{u}}{\partial \overline{\nu}} = -\frac{\partial \overline{u}}{\partial \overline{r}} = \overline{q}_0 := \overline{p}_0 + \sum_{k=1}^{N} \{\overline{p}_k \cos k\overline{\theta} + \overline{q}_k \sin k\overline{\theta}\} \quad \text{on } \partial S_{R_1}, \tag{2.7}$$

where $\overline{a}_k, \overline{b}_k, \overline{p}_k$ and $\overline{q}_k$ are coefficients. In (2.4)–(2.7), $\theta$ and $\overline{\theta}$ are the polar coordinates of $S_R$ and $S_{R_1}$ with the origins (0,0) and ($-R_1$,0), respectively, and $\nu$ and $\overline{\nu}$ are the outer normal of $\partial S_R$ and $\partial S_{R_1}$ respectively. For the Dirichlet, the Neumann conditions and their mixed types on $\partial S_R$ are given with the known coefficients.

In $S$, denote two nodes $\mathbf{x} = Q = (x,y) = (\rho,\theta)$, and $\mathbf{y} = P = (\xi,\eta) = (r,\phi)$, where $x = \rho\cos\theta, y = \rho\sin\theta, \xi = r\cos\phi$, and $\eta = r\sin\phi$. Then $\rho = \sqrt{x^2+y^2}$ and $r = \sqrt{\xi^2+\eta^2}$. The FS of Laplace's equation is given by $\ln \overline{PQ} = \ln\sqrt{\rho^2 - 2\rho r\cos(\theta-\phi) + r^2}$. From the BEM theory, we have different formulas for different locations of the field nodes $Q(\mathbf{x})$:

$$\int_{\partial S} \left\{ \ln|PQ| \frac{\partial u(\mathbf{y})}{\partial \nu} - u(\mathbf{y}) \frac{\partial \ln|PQ|}{\partial \nu} \right\} d\sigma_{\mathbf{y}} = \begin{cases} -2\pi u(Q), & Q \in S, \\ -\pi u(Q), & Q \in \partial S, \\ 0 & \text{otherwise}, \end{cases} \tag{2.8}$$

where $P(\mathbf{y}) \in (S \cup \partial S)$, and the series expansions of the FS $\ln|PQ|$ are given by (see [1,25,29,31])

$$\ln|PQ| = \ln|P(\mathbf{y}) - Q(\mathbf{x})| = \ln|P(r,\phi) - Q(\rho,\theta)|$$
$$= \begin{cases} U^i(\mathbf{x},\mathbf{y}) = \ln r - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{r}\right)^n \cos n(\theta-\phi), & \rho < r, \\ U^e(\mathbf{x},\mathbf{y}) = \ln \rho - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{\rho}\right)^n \cos n(\theta-\phi), & \rho > r, \end{cases} \tag{2.9}$$

where $\mathbf{x} = (\rho,\theta)$ and $\mathbf{y} = (r,\phi)$. Then we have two kinds of derivative expansions of FS,

$$\frac{\partial U^i(\mathbf{x},\mathbf{y})}{\partial r} = \frac{1}{r} + \sum_{n=1}^{\infty} \left(\frac{\rho^n}{r^{n+1}}\right) \cos n(\theta-\phi), \quad \rho < r, \tag{2.10}$$

$$\frac{\partial U^e(\mathbf{x},\mathbf{y})}{\partial r} = -\sum_{n=1}^{\infty} \left(\frac{r^{n-1}}{\rho^n}\right) \cos n(\theta-\phi), \quad \rho > r, \tag{2.11}$$

where the superscripts "e" and "i" designate the exterior and interior field nodes $\mathbf{x}$, respectively.

To distinguish the boundary element method (BEM) which is based on the second equation of the Green formula (2.8), the NFM is based on the third equation by using the FS expansions, we have

$$\int_{\partial S_R \cup \partial S_{R_1}} U(\mathbf{x},\mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial \nu}\, d\sigma_{\mathbf{y}} = \int_{\partial S_R \cup \partial S_{R_1}} u(\mathbf{y}) \frac{\partial U(\mathbf{x},\mathbf{y})}{\partial \nu}\, d\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \overline{S}^c, \tag{2.12}$$

where $\overline{S}^c$ is the complementary domain of $S \cup \partial S$. Substituting the Fourier expansions (2.9)–(2.11) into (2.12) yields the basic algorithms of NFM, where the exterior normal of $\partial S_{R_1}$ is given by $\partial U(\mathbf{x},\mathbf{y})/\partial \nu = -\partial U(\mathbf{x},\mathbf{y})/\partial r$. Although the basic descriptions above have been used in Chen's many papers, there exist no explicit equations reported so far. The explicit equations are important not only to understand the intrinsic nature of the NFM, but also to extend their applications. The first goal of this paper is to develop the explicit algebraic equations of the NFM.

In the Green formula (2.12), the field node $\mathbf{x} = (\rho,\theta)$ is supposed to locate outside of the solution domain $S \cup \partial S$ only; this is why the algorithms of Chen is called the null field method (NFM). The first question arises: Can we locate the field node just on the domain boundary: $\mathbf{x} \in \partial S$? If yes, *a puzzle* is encountered since $\mathbf{x} \in \partial S$ is not allowed in the first and the third equations on the right hand side of (2.8). Moreover, how to find a better choice of location of $\mathbf{x}$? To this end, we should choose suitable criteria to judge $\mathbf{x}$'s location. The most important criteria are errors and stability. Therefore, in this paper, important boundary errors are defined, the condition number and the effective condition number in [30] are chosen for illustrating of stability.

### 2.2. Explicit algebraic equations

First, consider the exterior field nodes $\mathbf{x} = (\rho,\theta)$ with $\rho > r = R$. On $\partial S_R$, by substituting (2.4)–(2.5) and (2.9)–(2.11) into (2.12), we have from orthogonality of Fourier series,

$$\int_{\partial S_R} u(\mathbf{y}) \frac{\partial U^e(\mathbf{x},\mathbf{y})}{\partial \nu}\, d\sigma_{\mathbf{y}} = \int_0^{2\pi} \left\{ a_0 + \sum_{k=1}^{M} [a_k \cos k\phi + b_k \sin k\phi] \right\}$$
$$\times \left\{ -\sum_{n=1}^{\infty} \left(\frac{R^{n-1}}{\rho^n}\right) \cos n(\theta-\phi) \right\} R\, d\phi$$

[1] The series expansions of FS is called the degenerate kernel of FS in Chen's publications.