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Numerical identification for impedance coefficient by a MFS-based optimization method ☆

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ABSTRACT

In this paper, we propose a new numerical method to solve an inverse impedance problem for Laplace's equation. The Robin coefficient in the impedance boundary condition is recovered from Cauchy data on a part of boundary. A crucial step is to transform the problem into an optimization problem based on the MFS and Tikhonov regularization. Then the popular conjugate gradient method is used to solve the minimization problem. We compare several stopping rules in the iteration procedure and try to find an accurate and stable approximation. Numerical results for four examples in 2D and 3D cases will show the effectiveness of the proposed method.

1. Introduction

We consider in this paper an inverse impedance problem, also called Robin inverse problem originating from corrosion detection in which the corrosion damage is represented by a Robin coefficient appearing in the third kind boundary condition on a portion of boundary. The identification of the unknown Robin coefficient is of great importance in engineering.

Let $\Omega \subset \mathbb{R}^d, d=2,3$ be a simply connected bounded domain with a piecewise smooth boundary $\partial\Omega$. Denote by n the outward unit normal to $\partial\Omega$ and assume $\partial\Omega = \overline{\Gamma}_c \cup \overline{\Gamma}_m$ where Γ_c and Γ_m are two open disjoint portions of $\partial\Omega$. The electrostatic potential or the steady temperature u satisfies the following boundary value problem

$$\triangle u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = f \quad \text{on } \Gamma_c, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_c, \tag{1.3}$$

$$\frac{\partial u}{\partial n} + \sigma u = h \quad \text{on } \Gamma_m, \tag{1.4}$$

where σ is a nonnegative function indicating the corrosion amount in some situations; f is the measured voltage or temperature and g is the measured current or heat flux on the accessible boundary portion Γ_c; h is a given function. The inverse problem we are concerned with is to determine the impedance function σ from the measured data f, g.

The uniqueness for the inverse impedance problem has been obtained by using the unique continuation property, refer to [5,7,13]. Various stability estimates have been investigated widely in the literature [1,4,5,7]. On numerical computations, we can find some references [2,3,14,16,13,19,25] for some of the most recent contributions. In [2,3], the authors provided two kinds of integral equation methods to solve the Cauchy problem (1.1)–(1.3) and then further obtain an approximate value to the impedance from (1.4). The calculation of singular integrals is required in the integral equation methods. In [14,16], the Robin inverse problem is formulated as a variational problem with an appropriate regularization, then the conjugate gradient method is employed to solve the variational problem in which the three direct mixed boundary value problems for Laplace's equation should be solved at each iteration step.

In this paper, based on the method of fundamental solutions (MFS), the potential is approximated by a linear combination of fundamental solutions whose singularities are located outside the solution domain. Both unknown coefficients in the linear combination and Robin coefficients will be determined by solving a nonlinear optimization problem which can be obtained by fitting the boundary data and adding two Tikhonov regularization terms. Then we use a conjugate gradient method to solve the function optimization problem. It can be noted that we do not need to calculate any quadratures and solve a direct problem.

The MFS has been used extensively for solving various direct problems of linear elliptic equations. Details can be found in the review papers of Fairweather and Karageorghis [9], Cho et al. [6]

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and Golberg and Chen [10] as well as further works after that. On the applications of the MFS to inverse problems for elliptic equations, we just mention the papers [22,24,23,20,15,27,18] for some of the contributions. To deal with inverse heat conduction problems, one can see [11,12,26,29,8,28,17] for instances. A brief review on the applications of the MFS to some inverse problems is issued in [21]. We note that the MFS is very effective to solve some linear inverse problems; however, very few papers are focused on a nonlinear ill-posed problem. In this paper we try to combine the MFS and the conjugate gradient method to solve the inverse impedance problem which is a severely ill-posed nonlinear inverse problem. Numerical examples for 2D and 3D cases will show that the proposed method is robust for the recovery of unknown impedance on both accuracy and stability.

The remainder of this paper contains three sections. In Section 2, we present the numerical method for dealing with the inverse impedance problem. In Section 3, three numerical examples in 2D and one in 3D are displayed to verify the efficiency and stability of the method. A brief conclusion is given in Section 4.

2. Numerical method based on the method of fundamental solutions

In this section, we propose a new numerical method based on the MFS for solving the inverse impedance problem.

The fundamental solution for the Laplace equation is

$$G(P,Q)=\begin{cases}-\dfrac{1}{2\pi}\ln\|P-Q\|, & d=2,\\ \dfrac{1}{4\pi\|P-Q\|}, & d=3,\end{cases} \tag{2.1}$$

where P, Q are points in $\mathcal{R}^d$ with $\|P-Q\|$ denoting their Euclidean distance. Let the source point Q be located outside the domain $\overline{\Omega}$, then the fundamental solution satisfies the Laplace equation in domain Ω. Based on the idea of the MFS, we can use a linear combination of fundamental solutions with source points chosen on a circle outside Ω to approximate the unknown function u. That is, we try to seek an approximate solution with the following form

$$\tilde{u}(P)=\sum_{j=1}^{n_s} c_j G(P,Q_j),\quad P\in\overline{\Omega}, \tag{2.2}$$

where Q_j are source points located outside $\overline{\Omega}$ and $\{c_j\}$ are unknown coefficients to be determined in the following minimization problem.

We choose collocation points $\{P_j\}_{j=1}^{n_c}$ on Γ_c for fitting the Cauchy data on boundary Γ_c and select some points $\{\overline{P}_j\}_{j=1}^{n_m}$ on Γ_m for matching the impedance condition (1.4). Then we can find the unknown coefficients c_j and the Robin coefficients σ_i by solving the following optimization problem:

$$\min_{c,\sigma} F(c,\sigma)=\frac{1}{2}\|A_1c-b_1\|^2+\frac{1}{2}\|A_2c-b_2\|^2+\frac{1}{2}\|(A_3+\Sigma A_4)c-b_3\|^2 +\frac{\mu}{2}\|c\|^2+\frac{\mu}{2}\|\sigma\|^2, \tag{2.3}$$

where $\mu>0$ is a regularization parameter; $c=(c_1,c_2,\ldots,c_{n_s})^T$, $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_{n_m})^T$ and

$$A_1=(G(P_i,Q_j))_{n_c\times n_s},\quad A_2=\left(\frac{\partial G}{\partial n}(P_i,Q_j)\right)_{n_c\times n_s},$$

$$A_3=\left(\frac{\partial G}{\partial n}(\overline{P}_k,Q_j)\right)_{n_m\times n_s},\quad A_4=(G(\overline{P}_k,Q_j))_{n_m\times n_s},$$

$$\Sigma=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_{n_m}),\quad \sigma_k=\sigma(\overline{P}_k),$$

as well as

$$b_1=(f(P_i))_{n_c\times 1},\quad b_2=(g(P_i))_{n_c\times 1},\quad b_3=(h(\overline{P}_k))_{n_m\times 1}. \tag{2.4}$$

By a simple calculation, we can obtain the gradients of function F to c and σ, respectively as follows:

$$F'_c(c,\sigma)=A_1^T(A_1c-b_1)+A_2^T(A_2c-b_1)+(A_3+\Sigma A_4)^T((A_3+\Sigma A_4)c-b_3)+\mu c,$$

$$F'_\sigma(c,\sigma)=((A_3+\Sigma A_4)c-b_3)^T(A_4c)+\mu\sigma,$$

where the superscript T represents the transpose of a matrix. Since the gradients of the function F is explicitly given, we can use a conjugate gradient algorithm to solve the minimization problem (2.3). In the following, the Fletcher–Reeves CGM applied to (2.3) is listed.

(1) Choose the initial guess $c^0=(0,0,\ldots,0)^T$, $\sigma^0=(0,0,\ldots,0)^T$ and calculate the initial descent directions $d_c^0=-F'_c(c^0,\sigma^0)$ and $d_\sigma^0=-F'_\sigma(c^0,\sigma^0)$. Set $k=0$.

(2) Update the coefficients vector c^{k+1} and the Robin coefficient vector σ^{k+1} by

$$c^{k+1}=c^k+\alpha_k d_c^k,\quad \sigma^{k+1}=\sigma^k+\alpha_k d_\sigma^k,$$

where the step length α_k can be given by a quadratic approximation of the function $F(c^k+\alpha d_c^k,\sigma^k+\alpha d_\sigma^k)$ to parameter α as follows:

$$\begin{aligned}\alpha_k=&-[(A_1c^k-b_1)^T(A_1d_c^k)+(A_2c^k-b_2)^T(A_2d_c^k)\\&+((A_3+\Sigma^kA_4)c^k-b_3)^T(D_\sigma^kA_4c^k+(A_3+\Sigma^kA_4)d_c^k)\\&+\mu(c^k)^Td_c^k+\mu(\sigma^k)^Td_\sigma^k]/w_k,\end{aligned}$$

$$\begin{aligned}w_k=&\|A_1d_c^k\|^2+\|A_2d_c^k\|^2+\|D_\sigma^kA_4c^k+(A_3+\Sigma^kA_4)d_c^k)\|^2\\&+\mu\|d_c^k\|^2+\mu\|d_\sigma^k\|^2\end{aligned}$$

with $D_\sigma^k=diag(d_\sigma^k)$.

(3) Update the descent direction by

$$d_c^{k+1}=-F'_c(c^{k+1},\sigma^{k+1})+\beta_kd_c^k,\quad d_\sigma^{k+1}=-F'_\sigma(c^{k+1},\sigma^{k+1})+\beta_kd_\sigma^k,$$

where the conjugate coefficient β_k is given by

$$\beta_k=\frac{\|F'_c(c^{k+1},\sigma^{k+1})\|^2+\|F'_\sigma(c^{k+1},\sigma^{k+1})\|^2}{\|F'_c(c^k,\sigma^k)\|^2+\|F'_\sigma(c^k,\sigma^k)\|^2}.$$

(4) Increase k by one and go to Step (2). Repeat the above procedure until a fixed number.

3. Numerical experiments

In this section, we present some numerical results for three examples in 2D and one example in 3D to show the feasibility of the proposed method.

In all the computations, the noisy data are generated by

$$f^\delta(P_i)=f(P_i)(1+\varepsilon_r\,\mathrm{rand}(i)),\quad i=1,2,\ldots,n_c,$$

and

$$g^\delta(P_i)=g(P_i)(1+\varepsilon_r\,\mathrm{rand}(i)),\quad i=1,2,\ldots,n_c,$$

where $f(P_i)$, $g(P_i)$ are the exact Cauchy data at point P_i; the value ε_r is a relative noise level and the function rand(i) are random numbers uniformly distributed in $[-1,1]$. The noise level in L^2 norm is given approximately by

$$\delta:=\left\{\frac{1}{2n_c}\sum_{i=1}^{n_c}[(f^\delta(P_i)-f(P_i))^2+(g^\delta(P_i)-g(P_i))^2]\right\}^{1/2}. \tag{3.1}$$