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Boundary element formulation of axisymmetric problems for an elastic halfspace

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ABSTRACT

Axisymmetric problems for an elastic halfspace are commonly analyzed by the boundary element (BE) method by employing the axisymmetric fundamental solution for the fullspace. In such cases, the discretization of the free surface is required, with its truncation at an appropriate location from the axis of symmetry. This paper presents the BE implementation of the axisymmetric fundamental solution for an elastic halfspace, given in terms of integrals of the Lipschitz–Hankel type, that satisfies in advance the boundary condition of zero traction on the free surface and the decay of displacements in the far field. Explicit equations for post-processing the results at internal points are provided, as well as adequate numerical schemes to evaluate the boundary integrals arising in the method. This formulation can be easily implemented in existing BE computational codes for axisymmetric fullspace problems, requiring only a few modifications. Numerical results are provided to validate the proposed formulation.

1. Introduction

The axisymmetric formulation in classical elasticity is useful for the analysis of problems in geomechanics [1,2], as well as contact problems for cylinders, spheres and circular plates [3–8]. Other applications involve the study of fracture mechanics phenomena and inclusions [5,9–11].

In particular, the BE method is advantageous for axisymmetric problems, since it reduces the analysis of the three-dimensional domain to a one-dimensional mesh discretization requiring only the evaluation of linear integrals. However, the fundamental solutions involved are more complex, requiring special considerations on their manipulation and integration to correctly evaluate the influence coefficients arising in the boundary integral equations. Extensive surveys on the existing axisymmetric fundamental solutions are given by Wang and Liao [12,13], Wang et al. [14] and Wideberg and Benitez [15].

The BE method for axisymmetric elasticity was first formulated by Cruse et al. [16], using the fullspace fundamental solution derived by Kermanidis [17]. Several contributions to the formulation of the axisymmetric problem may be cited, such as the expansion of non-symmetric boundary conditions by Fourier series suggested by Mayr [18] and Rizzo and Shippy [19,20], and the assessment of body forces by means of particular integrals incorporated by Park [21]. Also, axisymmetric formulations have been developed for transverse isotropy [22], thermoelasticity [23], elastoplasticity [24] and viscoplasticity [25]. In elastodynamics, the works by Wang and Banerjee [26,27], Tsinopoulos et al. [28] and Yang and Zhou [29] in the frequency domain should be mentioned. The method has also been successfully applied to contact problems [30] and fracture mechanics [31].

For axisymmetric halfspace problems, the BE formulation employed with the fullspace fundamental solution requires the discretization of the infinite free surface. In this case, the surface must be truncated at a reasonable distance from the axis of symmetry and the region of interest [32]. The disadvantage of such a scheme is that a large number of boundary elements is needed to model the remote boundary satisfactorily, so that relative displacements in particular can be accurately evaluated.

An alternative way to deal with this problem is to use infinite boundary elements, as suggested by Watson [33]. These infinite elements, which simulate the decay of the displacements and stresses in the far field, are mapped onto a finite region in terms of an intrinsic coordinate system to facilitate the integration. A variety of infinite elements can be found in the literature for three-dimensional elasticity, depending on the mapping scheme used and the application [34–36]. However, such elements are not available for problems with axisymmetry, probably because treating the integration of the singular kernels over the mapped

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infinite elements is not straightforward for the fullspace fundamental solution. Therefore, Kelvin's fundamental solution is usually employed together with the available three-dimensional surface infinite elements for axisymmetric applications in the halfspace [37–39], thus requiring the boundary surfaces to be discretized.

Another way to treat this problem is to implement the fundamental solution that satisfies in advance the traction free boundary condition on the free surface, which circumvents its numerical discretization. In elasticity, this approach was used by Telles and Brebbia [40] and Dumir and Mehta [41] to examine problems for an isotropic and orthotropic halfplane, respectively.

This work presents a BE formulation for axisymmetric elasticity problems for a halfspace [42] that makes use of the fundamental solutions due to radial and axial ring loads embedded in a halfspace derived by Hasegawa [43,44]. The resulting equations could be manipulated by expressing the fundamental solutions in terms of Lipschitz–Hankel integrals, as adopted by Selvadurai and Rajapakse [5] using extensions to the solutions developed by Mindlin [45] and Mindlin and Cheng [46]. Since the terms of the fullspace fundamental solution can be identified as constituents of the halfspace fundamental solution, the proposed formulation can be implemented by introducing only a few modifications in existing axisymmetric computational codes. Explicit equations are presented for expressing results at internal points as well as appropriate numerical schemes to accurately evaluate the integrals arising in the formulation. Problems related to torsional loads, not addressed in this work, involve simpler fundamental solutions and can be examined in a similar manner.

Section 2 of this paper introduces the axisymmetric fundamental solution for the elastic fullspace and an elastic halfspace. Section 3 presents the axisymmetric BE formulation, followed by Section 4 that deals with the numerical integration. Finally, Section 5 illustrates numerical examples that validate the proposed formulation.

2. Axisymmetric fundamental solution

The axisymmetric fundamental solution for elasticity consists of displacements u^*_{ij}(P,Q) and stresses σ^*_{ijk}(P,Q) due to ring loads in the i-direction applied at P(ξ,z') and centered in the z-axis. The continuum has shear modulus μ and Poisson's ratio ν. The solutions are given in the cylindrical coordinate system (r,z). The indices j and k stand for the displacement and stress components measured at Q(r,z).

For the fullspace, displacements due to ring loads were first derived by Kermanidis [17], by applying Betti's theorem to the Papkovich–Neuber solution [47] for an elastic medium of infinite extent. Subsequently, Cruse et al. [16] and Bakr and Fenner [23] solved Navier's equilibrium equations by expressing the displacements as Galerkin vectors [47] and considering ring loads as body forces. Also, Shippy et al. [48] integrated Kelvin's solution [47] for the three-dimensional infinite medium along a circular path centered on the axis of symmetry.

For the halfspace, Hasegawa [43,44] deduced displacements and stresses from stress functions [49] obtained by means of Fourier and Hankel transforms and considering ring loads as body forces. Later, Selvadurai and Rajapakse [5] imposed boundary conditions and continuity conditions to displacements and stresses expressed by Muki's solution [50,51] and arrived at the same solutions. These solutions were also obtained by Hanson and Wang [52] as a particular case of the problem for the medium with transverse isotropy.

Both axisymmetric fundamental solutions for fullspace and halfspace can be expressed by means of either integrals of the Lipschitz–Hankel type involving products of Bessel functions [53], or complete elliptic integrals of the first and second types [54], or Legendre functions [54]. In this work, the approach presented by Selvadurai and Rajapakse [5] is adopted. Expressions are written in terms of integrals of the Lipschitz–Hankel type [53]

$$I_{pq\lambda}(\xi,r;c)=\int_0^\infty J_p(\xi t)J_q(rt)e^{-ct}t^\lambda\,dt \tag{1}$$

in which p, q and λ are integers, $J_p(\xi t)$ and $J_q(rt)$ are Bessel functions of the first kind of order p and q, respectively. The integrals arising in the axisymmetric fundamental solutions are convergent [53] and their closed form expressions in terms of complete elliptic integrals of the first, second and third kinds [54] are listed in Appendix A.

2.1. Ring loads in an elastic fullspace

The fundamental solution can be derived from Muki's solution [50,51] of the Navier equilibrium equations for an elastic isotropic medium,

$$(1-2\nu)\left(\nabla^2 u_r-\frac{u_r}{r^2}\right)+\Delta_{,r}=0 \tag{2}$$

$$(1-2\nu)\nabla^2 u_z+\Delta_{,z}=0 \tag{3}$$

where

$$\Delta=u_{r,r}+\frac{u_r}{r}+u_{z,z} \tag{4}$$

Muki represented displacements by means of harmonic and biharmonic functions and used Hankel transforms and their correspondence to generalized Fourier–Bessel transforms to arrive at a general asymmetric solution. This solution can be specialized for axisymmetry, leading to

$$u_r=\frac{1}{2}\int_0^\infty\left(\frac{dG}{dz}+2H\right)[J_1(rt)-J_{-1}(rt)]t^2\,dt \tag{5}$$

$$u_z=\int_0^\infty\left[(1-2\nu)\frac{d^2G}{dz^2}-2(1-\nu)t^2G\right]J_0(rt)\,dt \tag{6}$$

where

$$G(t,z)=(A+Bz)e^{zt}+(C+Dz)e^{-zt} \tag{7}$$

$$H(t,z)=Ee^{zt}+Fe^{-zt} \tag{8}$$

in which $A(t),B(t),\ldots,F(t)$ are unknown functions.

Consider a fullspace split into two parts, I and II, by a plane normal to z at $z=z'$ as shown in Fig. 1. Applying Eqs. (5) and (6) and the regularity conditions for the displacements and stresses as $z\to\pm\infty$,

$$u_i^{\mathrm{I,II}}(r,\pm\infty)=0,\quad \sigma_{ij}^{\mathrm{I,II}}(r,\pm\infty)=0 \tag{9}$$

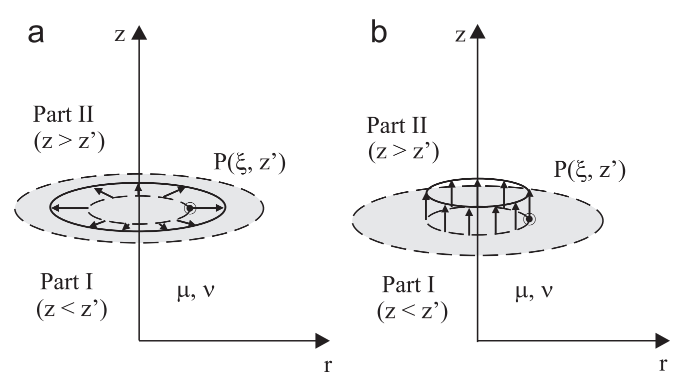

Fig. 1. Ring loads in the elastic fullspace: (a) radial direction; (b) axial direction.