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Solving the nonlinear Poisson-type problems with F-Trefftz hybrid finite element model

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ABSTRACT

A hybrid finite element model based on F-Trefftz kernels (fundamental solutions) is formulated for analyzing Dirichlet problems associated with two-dimensional nonlinear Poisson-type equations including nonlinear Poisson–Boltzmann equation and diffusion–reaction equation. The nonlinear force term in the Poisson-type equation is frozen by introducing the imaginary terms at each Picard iteration step, and then the induced Poisson problem is solved by the present hybrid finite element model involving element boundary integrals only, coupling with the particular solution method with radial basis function interpolation. The numerical accuracy of the present method is investigated by numerical experiments for problems with complex geometry and various nonlinear force functions.

1. Introduction

The nonlinear Poisson-type equation written by

$$\frac{\partial^2 u}{\partial X_1^2}+\frac{\partial^2 u}{\partial X_2^2}=f(X_1,X_2,u) \text{ in } \Omega\in\mathbb{R}^2 \tag{1}$$

is the governing equations for many physical problems including heat conduction, mass transfer, seepage, electric and magnetic fields, and chemical reaction. In the nonlinear equation mentioned above, $u(X_1,X_2)$ is the scalar potential at a given field point (X_1,X_2), f is a generalized force function related to the unknown potential u, and Ω is the two dimensional bounded domain of the problem under consideration.

The complete boundary value problems should consist of the governing equation (1) and the following Dirichlet boundary conditions:

$$u=\overline{u} \text{ on } \Gamma_u \tag{2}$$

where an over bar denotes the specified potential.

Owing to the nonlinear behavior of the partial differentiable equation (PDE) under consideration, the theoretical analysis has been proved to be considerably difficult, especially for problems with irregular geometries and non-uniform boundary conditions. To study the nonlinearity in complex solution domain, it is a long history in resorting to numerical solutions. So far, different numerical techniques, including finite element method (FEM) [1], boundary element method (BEM) [2–4], finite difference method (FDM) [5], the hybrid Trefftz finite element method (HT-FEM) [6], the method of fundamental solution (MFS) [7–11], the locally boundary integral equation (LBIE) method [12], Kansa method with radial basis function (RBF) interpolation [13,14], were developed for solving the nonlinear Poisson-type potential problems consisting of Eqs. (1) and (2). Among them, the HT-FEM has recently been paid more attention in the past decades [6,15–18], because it has some advantages over the conventional FEM and boundary element method (BEM). For example, it possesses versatile element construction and can capture the variation of singular fields or stress concentration by employing suitable interpolation kernels. More discussion on this topic can be found in literatures [17,18].

As an alternative to the HT-FEM, the hybrid finite formulation with F-Trefftz functions or fundamental solutions as interpolating kernel functions within the elements has been established for analyzing heat conduction and elastic problems [19–21]. The method was known as HFS-FEM for distinguishing the presented model from HT-FEM. In the HFS-FEM, the fundamental solutions (F-Trefftz functions), instead of T-complete functions (Trefftz functions), are used to construct the interior field, and independent boundary frame field is approximated by conventional shape functions. A new variational functional was constructed to guarantee the inter-element continuity, link the two fields and establish the final force–displacement equations. The use of fundamental solutions can convert the domain integral in the variational functional to element boundary integrals. It is worth

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noting that no singular integrals are involved in the HFS-FEM, although the fundamental solutions are employed. It is because the source points used for the evaluation of fundamental solution are placed outside the element of interest as was done in the MFS. Thus, the source point and field point never overlap during the computation. Clearly, the present HFS-FEM inherits the advantages of HT-FEM and simultaneously can alleviate or remove some drawbacks of HT-FEM such as the properly selection of the number of terms of T-complete functions, the complicated coordinate transformation needed in the HT-FEM, and the relatively complicated expressions by comparison with the fundamental solutions which usually contains one term only, rather than a series containing infinite terms, and are available for most physical problems.

In this paper, we focus on the extension of the developed HFS-FEM to the nonlinear Poisson-type problems. The nonlinear term appearing in the right-hand side of the differential equation is first frozen by introducing the Picard iteration process, and then in each iteration step, radial basis functions are employed to approximate the part of particular solutions, and the HFS-FEM is formulated to determine the homogeneous potential distribution with modified boundary conditions. Both the iteration residual and the inter-iteration difference are used to assess the convergent performance.

The paper is arranged as follows. The solution procedure including the iterative method and the presented hybrid finite element formulation is stated in Section 2. In Section 3, we consider some numerical examples including nonlinear Poisson–Boltzmann equation and nonlinear diffusion–reaction equation. A comparison of the numerical results from HFS-FEM is made with those from either analytical approach or other numerical methods. Finally, some conclusions are presented in Section 4.

2. Solution methodology

In order to develop a generalized algorithm for the nonlinear boundary value problem (BVP) consisting of Eq. (1) and Dirichlet boundary condition (2), the Picard method of iteration implemented by Chen et al. [11] for the two-dimensional Dirichlet problem is employed for solving nonlinear Poisson-type equations with HFS-FEM. the desired solutions are obtained by assuming u at the start of the iterations and solving the linearizing equation at each level of iteration. For this purpose, we construct the following linear iteration process to evaluate the potential distribution at the current iteration step (m) by freezing the nonlinear term appeared in the right-hand term f which is evaluated using the results at previous iteration $(m-1)$ step

$$\frac{\partial^2 u^{(m)}}{\partial X_1^2}+\frac{\partial^2 u^{(m)}}{\partial X_2^2}=f(X_1,X_2,u^{(m-1)}) \quad \text{in } \Omega\in\mathbb{R}^2 \tag{3}$$

It is obvious that the sequence $\{u^{(m)}\}$ is expressed in terms of $\{u^{(m-1)}\}$, which is known from previous iteration step, for each $m(m=1,2,3,\ldots)$, and $u^{(0)}$ represents any initial guess. To complete the iteration and obtain convergent results, the iteration convergent criterion is set by controlling both the maximum residual related to the nonlinear governing equation

$$\left\|\frac{\partial^2 u^{(m)}}{\partial X_1^2}+\frac{\partial^2 u^{(m)}}{\partial X_2^2}-f(X_1,X_2,u^{(m)})\right\|_\infty \le \varepsilon_1 \tag{4}$$

and the inter-iteration difference

$$\left\|u^{(m)}-u^{(m-1)}\right\|_\infty \le \varepsilon_2 \tag{5}$$

where $\|\bullet\|_\infty$ represents the infinite norm, and ε_1 and ε_2 are iteration tolerances, respectively.

Generally, the treatment of nonhomogeneous term f involves the domain integral. To remove the domain integral from the element stiffness equation, the radial basis functions are employed in this work.

Before introducing radial basis functions, it is observed that the linearity of Eq. (3) makes its solution to be divided into two major parts

$$u=u_h+u_p \tag{6}$$

where the nonhomogeneous solution, also named as particular solution, u_p is required to satisfy

$$\nabla^2 u_p=f(X_1,X_2,u^{(m-1)}) \tag{7}$$

without any boundary conditions, and the homogeneous solution u_h is obtained by solving the following linear system with modified boundary condition:

$$\begin{cases}\nabla^2 u_h=0 & \text{in } \Omega\\ u_h=\overline{u}-u_p & \text{on } \Gamma_u\end{cases} \tag{8}$$

2.1. Particular solution

In order to obtain the particular solution, the radial basis functions are used here to approximate the induced fictitious function f, that is

$$f(X_1,X_2,u^{(m-1)})=\sum_{k=1}^{N_I}\alpha_k\varphi_k(X_1,X_2)=\{\boldsymbol{\varphi}\}\{\boldsymbol{\alpha}\} \tag{9}$$

where N_I denotes the number of interpolation points in the domain of interest, α_k are unknown interpolating coefficients, $\varphi_k(X_1,X_2)$ be radial basis function centered at the point (X_1^k,X_2^k), and $\{\boldsymbol{\varphi}\}=\{\,\varphi_1\quad\varphi_2\quad\cdots\quad\varphi_{N_I}\,\}$, $\{\boldsymbol{\alpha}\}=\{\,\alpha_1\quad\alpha_2\quad\cdots\quad\alpha_{N_I}\,\}^T$ are corresponding basis vector and coefficient vector, respectively.

Radial basis functions are usually expressed in terms of the Euclidian distance, so they can work well in any dimensional space and does not increase the computational difficulty when the dimension of a problem increases. In most numerical analyses, the commonly used RBFs include linear polynomial, thin plate spline (TPS) and multiquadric (MQ). Among them, the linear polynomial and thin plate spline are piecewise smooth in the space, while the MQ is infinitely smooth. Due to the high sensitivity to shape parameter in MQ, we will not employ MQ in the paper, although it can achieve good accuracy in some cases.

For the case of thin plate spline (TPS) basis, the interpolation kernel has the following expression:

$$\varphi_k(X_1,X_2)=r^2\ln r \tag{10}$$

where r represents the Euclidean distance of the given point (X_1,X_2) from a fixed point (X_1^k,X_2^k) in the domain of interest

$$r=\sqrt{(X_1-X_1^k)^2+(X_2-X_2^k)^2} \tag{11}$$

In the standard dual reciprocity procedure, it is reasonable to assume that the particular solution is approximated by

$$u_p(X_1,X_2)=\sum_{k=1}^{N_I}\alpha_k\psi_k(X_1,X_2)=\{\boldsymbol{\psi}\}\{\boldsymbol{\alpha}\} \tag{12}$$

so that a relationship between the basis φ_k and the particular kernel ψ_k

$$\nabla^2\psi_k(X_1,X_2)=\varphi_k(X_1,X_2) \tag{13}$$

exists. In Eq. (13), $\{\boldsymbol{\psi}\}=\{\,\psi_1\quad\psi_2\quad\cdots\quad\psi_{N_I}\,\}$ represents the set of approximated particular kernel.