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On multiplicative consistency of interval and fuzzy reciprocal preference relations

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ABSTRACT

Extension of Saaty's definition of consistency to interval and fuzzy reciprocal preference relations is studied in the paper. The extensions of the definition to interval and triangular reciprocal preference relations proposed by Wang (2005), Liu (2009), Liu et al. (2014) and Wang (2015a, 2015b) are reviewed and some shortcomings in the definitions are pointed out. Particularly, as was already shown by Wang (2015a, 2015b), the definitions of consistency proposed by Liu (2009) and Liu et al. (2014) are not invariant under permutation of compared objects. Wang's (2015a, 2015b) definitions rectify this drawbacks. However, as is pointed out in this paper, Wang's definitions of consistent interval and triangular reciprocal preference relations do not keep the reciprocity of pairwise comparisons, which is the substance of reciprocal preference relations. In this paper, definitions of consistent interval, triangular and trapezoidal reciprocal preference relations invariant under permutation of compared objects and preserving the reciprocity of pairwise comparisons are introduced. Useful tools for verifying the consistency are proposed and some properties of consistent interval and fuzzy reciprocal preference relations are derived. Furthermore, the new definition of consistency for interval reciprocal preference relations is compared with the definition of consistency proposed by Wang et al. (2005), and numerical examples are provided to illustrate the difference between the consistency definitions.

1. Introduction

Reciprocal preference relations play a significant role in multi-criteria decision-making methods based on pairwise comparisons of objects. The most known method using reciprocal preference relations is Analytic Hierarchy Process (AHP in the following) developed by Saaty (1977, 1980). In this method, Saaty defined a scale of integers from 1 to 9 with assigned linguistic terms expressing the intensity of preference of one compared object over another one. Using this scale, reciprocal preference relations of alternatives and criteria are constructed. By applying various methods, priorities of alternatives and criteria are then elicited from the reciprocal preference relations and finally aggregated into overall priorities of alternatives.

In practical applications of reciprocal preference relations, concept of consistency plays an important role. In particular, in order to guarantee that the priorities of objects derived from reciprocal preference relations are reasonable, consistency of the preference information provided in the reciprocal preference relations should be verified. For that, Saaty (1977) provided a definition of consistent reciprocal preference relations based on the multiplicative-transitivity property. Later, also other definitions of consistency were proposed; see, e.g., Basile and D'Apuzzo (2002) and Stoklasa, Jandová, and Talašová (2013). Because not always reciprocal preference relations are consistent, various indices for measuring acceptable inconsistency of reciprocal preference relations have also been proposed.

Nevertheless, reciprocal preference relations are not able to handle the imprecision of information in real decision-making problems. Furthermore, crisp numbers representing linguistic terms expressing the intensity of preference of one compared object over another one cannot handle the vagueness in their meaning. For these reasons, the extension of the AHP method and reciprocal preference relations to intervals and fuzzy numbers has been studied; see, e.g., Laarhoven and Pedrycz (1983), Buckley (1985), Cheng and Mon (1994), Chang (1996), Xu (2000), Buckley, Feuring, and Hayashi (2001), Csutora and Buckley (2001), Enea and Piazza (2004), Krejčí, Pavlačka, and Talašová (2017), and Krejčí (2016, 2017a, 2017b, 2017c).

It should be mentioned here that harsh critics of fuzzy extension of AHP appeared recently, and fallacy of all well-known fuzzy AHP methods was claimed by Zhü (2014). However, shortly

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afterwards, Fedrizzi and Krejčí (2015) demonstrated that these critics are not well-founded as they are based on arguments contradicting commonly accepted results of fuzzy set theory. Furthermore, Fedrizzi and Krejčí (2015) showed that it is possible to extend the AHP methods to fuzzy numbers properly by applying the constrained fuzzy arithmetic and preserving the reciprocity of pairwise comparisons, which is the substance of reciprocal preference relations.

Also in interval and fuzzy reciprocal preference relations, consistency of preference information plays a very important role since the inconsistency can lead to wrong decisions. That is the reason why consistency of interval and fuzzy reciprocal preference relations and measures of inconsistency have been studied extensively; see, e.g., Buckley (1985), Wang, Yang, and Xu (2005), Liu (2009), Liu, Zhang, and Zhang (2014), Krejčí and Stoklasa (2016), Krejčí (2015, 2017a, 2017b, 2017c), Zheng, Zhu, Tian, Chen, and Sun (2012), Gavalec, Ramík, and Zimmermann (2014), Li, Wang, and Tong (2016), and Jandová, Krejčí, Stoklasa, and Fedrizzi (2016).

Definitions of consistency and inconsistency indices for interval and fuzzy reciprocal preference relations should preserve two basic properties - invariance under permutation of objects and reciprocity of pairwise comparisons. According to Brunelli and Fedrizzi (2015), invariance under permutation of objects is a desirable property. In fact, they introduced this property as one of the axioms characterizing inconsistency indices. Moreover, the lack of invariance under permutation of objects of some definitions of consistency for interval fuzzy preference relations was already pointed out and criticized by Wang (2014), Wang and Chen (2014), and Krejčí (2017b, 2017c). Therefore, the definition of consistency for interval and fuzzy reciprocal preference relations, similarly as Saaty's definition of consistency for reciprocal preference relations, should not depend on permutation of objects compared in the preference relation.

Similarly, reciprocity of pairwise comparisons is an inherent property of reciprocal preference relations which needs to be extended properly also to interval and fuzzy reciprocal preference relations. This does not concern only the simple reciprocity of corresponding intervals and fuzzy numbers in the interval and fuzzy reciprocal preference relations, respectively. As emphasized, e.g., by Fedrizzi and Krejčí (2015), the constrained fuzzy arithmetic introduced by Klir and Pan (1998) needs to be employed in the extension of reciprocal preference relations to fuzzy numbers and intervals in order to handle properly the reciprocity property.

With extension of reciprocal preference relations to intervals and fuzzy numbers, also another interesting issue emerges. Pairwise comparisons provided by decision makers in interval and fuzzy reciprocal preference relations can be highly indeterminate - the corresponding intervals or fuzzy numbers can be very vague. This may lead to highly indeterminate results (interval or fuzzy priorities of objects) with just a little information useful for making a decision. Therefore, it might be useful to measure the indeterminacy of interval and fuzzy reciprocal preference relations. An interesting approach for measuring indeterminacy of interval reciprocal preference relations was introduced by Li et al. (2016).

In this paper, the extension of the original definition of consistency for reciprocal preference relations given by Saaty (1977) (that is the consistency based on the multiplicative-transitivity property) is focused on. The extensions of the definition proposed in the literature are reviewed and drawbacks of some of them are pointed out. The extension of the definition to interval reciprocal preference relations proposed by Liu (2009) and the fuzzy extension to triangular and trapezoidal reciprocal preference relations proposed by Liu et al. (2014) are reviewed briefly. As was already pointed out by Wang (2015a, 2015b), the definitions proposed by Liu (2009) and Liu et al. (2014) are dependent on the labeling of objects. Wang (2015a, 2015b) proposed other definitions of consistency for interval and triangular reciprocal preference relations. These definitions of consistency are based on the extension of a property equivalent to Saaty's definition of consistency of reciprocal preference relations. However, it is shown in this paper that the extension of this property is not done properly as it is based on the standard interval fuzzy arithmetic and thus it violates the reciprocity of pairwise comparisons in interval and fuzzy reciprocal preference relations. Afterwards, a proper fuzzy extension of the definition of consistency given by Saaty (1977) is proposed. Properties of consistent interval and fuzzy reciprocal preference relations are studied, and the definition of consistent interval reciprocal preference relations is compared with the definition proposed by Wang et al. (2005). The definition of consistency proposed in this paper can be easily modified in order to be applied to fuzzy reciprocal preference relations with an arbitrary type of fuzzy numbers described uniquely by their $\alpha-$cuts.

The paper is organized as follows. In Section 2, basic notions of reciprocal preference relations, triangular and trapezoidal fuzzy numbers, standard and constrained fuzzy arithmetic, and interval and fuzzy reciprocal preference relations are given. In Section 3, the definitions of consistency proposed by Wang et al. (2005), Liu (2009), Liu et al. (2014), and Wang (2015a, 2015b) are reviewed. Further, the shortcomings of the definitions proposed by Liu (2009), Liu et al. (2014), and Wang (2015a, 2015b) are discussed. In Section 4, new definitions of consistency for interval, triangular and trapezoidal reciprocal preference relations are provided. In Section 5, the properties of consistent interval and fuzzy reciprocal preference relations are studied, and the comparison with the definition of consistent interval reciprocal preference relations proposed by Wang et al. (2005) is done. Finally, in Section 6, illustrative examples are provided, and the conclusion is done in Section 7.

2. Preliminaries

In this section, basic notions of reciprocal preference relations, triangular and trapezoidal fuzzy numbers, standard and constrained fuzzy arithmetic, and interval and fuzzy reciprocal preference relations are given.

2.1. Reciprocal preference relations

Reciprocal preference relations have their origins in AHP (Saaty, 1977, 1980). A reciprocal preference relation on a finite set of n objects $o_1, \ldots, o_n$ is represented by a square matrix $A = \{a_{ij}\}_{i,j=1}^{n}$. Element a_{ij} of the matrix represents the intensity of preference of object o_i over object o_j by means of the ratio of their priorities. In AHP, Saaty's scale of integer numbers 1–9 together with their reciprocals is usually used for expressing the intensities of preference on pairs of compared objects. To each element of the scale a linguistic term expressing the intensity of preference of one compared object over another one is assigned, see Table 1. A reciprocal preference

Table 1
Saaty's scale.

Intensity of preference	Linguistic term
1	Equal preference
3	Moderate preference
5	Strong preference
7	Very strong preference
9	Extreme preference
2, 4, 6, 8	Intermediate values between the two adjacent judgments connected by word "between"